Universiteit Utrecht

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Classifying Vector Bundles

Bachelor Thesis

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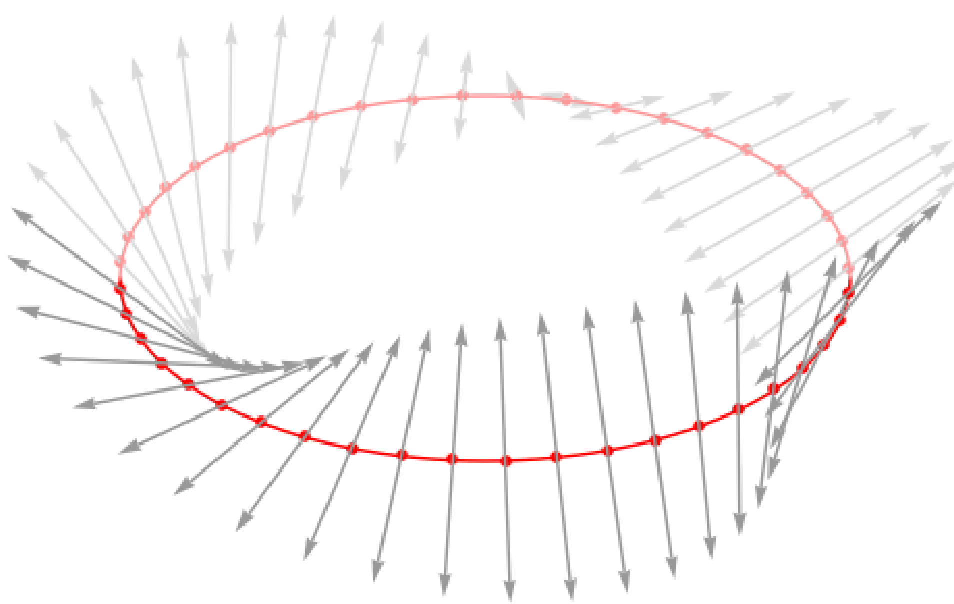
Mathematics



The Möbius bundle [Ega11]

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Introduction

At the end of this thesis, we will know the fundamental group of $GL_n(\mathbb{C})$, $\pi_1(GL_n(\mathbb{C})) \cong \mathbb{Z}$. However, the path we take to obtain this result is not the classical one.

Throughout this thesis, we will be studying vector bundles, an object in algebraic and differential geometry. Two of the simplest vector bundles are given by the Möbius band and the annulus. One is the twisted product of a line and a circle, the other the actual product of the two. More generally, vector bundles consist of a base space, in the particular case of the Möbius band and the annulus this is the circle, and a vector space attached to each point of the base space called the fiber, which in our particular case is the line. Intuitively it is clear the Möbius band and the annulus are two different objects, and part of this thesis is dedicated to developing tools to distinguish the two. These tools will not only be applicable to the Möbius band and the annulus, but to other vector bundles as well.

Our main interest will be vector bundles which have the n-dimensional sphere as their base space. The main question will be: "can we classify them?" To answer this question we will need and develop tools from algebraic topology, with the main result being Theorem 2.2.5. This theorem states the vector bundles over the n-sphere S^n can be classified using homotopy classes of maps $f\colon S^{n-1} \to GL_n(\mathbb{K})$.

Lastly, we will restrict our attention to vector bundles over the complex projective line $\mathbb{C}\mathrm{P}^1$. In 1957 Grothendieck showed one can classify holomorphic vector bundles over $\mathbb{C}\mathrm{P}^1$ by ordered sets of integers [Gro57], and we will give a similar result for topological vector bundles using techniques similar to [HM82]. We will be using the identification $\mathbb{C}\mathrm{P}^1 \cong S^2$ to be able to transfer our tools from vector bundles over spheres to vector bundles over $\mathbb{C}\mathrm{P}^1$, and this in turn will lead us to Theorem 3.4.4 and the identification $\pi_1(GL_n(\mathbb{C})) \cong \mathbb{Z}$.

This thesis is aimed at bachelor students that have taken introductory courses in algebraic topology, linear algebra and complex analysis. Most of this thesis is based on the first chapter of [Hat03], with much elaboration provided, and by the end one should have a foundation to study K-Theory as found in the second chapter of [Hat03].

I wish to thank Jack Davies for taking on my request of supervising my thesis, even though he had no obligation to do so. He has been a big help in providing interesting subjects and adequate resources, and has been very generous with his time and knowledge. I also wish to thank Lennart Meier for providing valuable input and feedback and providing the opportunity to work with Jack as my supervisor.

Conventions

- The natural numbers start at zero; $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.
- When writing $\mathbb{K}$, we mean either the field $\mathbb{R}$ or $\mathbb{C}$.
- With I we mean the closed unit interval $[0, 1]$.
- All maps and functions in this thesis are assumed to be continuous.
- All spaces discussed in this thesis are assumed to be Hausdorff.
- When writing $X \cong Y$ we mean the objects X and Y are isomorphic (homeomorphic, isomorphic as vector bundles, isomorphic as groups, etc.).
- When writing $f \simeq g$ or $X \simeq Y$ we mean two functions f and g are homotopic or two spaces X and Y are homotopy equivalent.

1 Vector Bundles

In this section we will be discussing the notion of a vector bundle, some examples, how to endow vector bundles with additional structure and how to construct new vector bundles out of existing ones. We will mainly be following the discussion in [Hat03], making a few sidesteps and elaborations as found in [MS75] and [Zin10].

1.1 Definitions and examples

Let us start with a motivating example: the tangent space of a manifold, in particular S^2. When thinking of the tangent space of S^2, we usually have the notion of attaching an individual copy of the vector space $\mathbb{R}^2$ to each individual point of $x \in S^2$, parallel to the surface of S^2 at that point. Here we view $S^2 \subset \mathbb{R}^3$ as embedded inside $\mathbb{R}^3$ as all points with unit distance from the origin. If we consider all the vector spaces attached to S^2, this becomes a family of vector spaces parameterized by points $x \in S^2$. This family is parameterized in a continuous way, and can be endowed with a topology. If we do so, we get the topological space TS^2, which is the disjoint union of all vector spaces corresponding to individual points on S^2, together with a topology.

One can ask if TS^2 is just $S^2 \times \mathbb{R}^2$ in disguise, since we are attaching a copy of $\mathbb{R}^2$ to every point of S^2. To be more precise, does there exist a homeomorphism $h\colon TS^2 \to S^2 \times \mathbb{R}^2$ taking each vector space in TS^2 to the corresponding vector space $\{x\} \times \mathbb{R}^2$ by linear isomorphism? In other words, does there exist a map from TS^2 to $S^2 \times \mathbb{R}^2$ respecting the global topological structure and the local linear structure? If there would exist such a map, we would be able to find a non-vanishing vector field on S^2, by taking a fixed non-zero $v \in \mathbb{R}^2$ and considering the vector field $\{h^{-1}(x, v)\}_{x \in S^2}$. This is known to be impossible by the *Hairy Ball Theorem* ([Hat03], §2.2), which implies TS^2 is distinguishable from $S^2 \times \mathbb{R}^2$ globally.

We do not need to restrict ourselves to the sphere, and we could ask this question for any (real) manifold M. We know the manifold locally looks like an open $U \subseteq \mathbb{R}^n$, and if we consider the tangent space this locally looks like $U \times \mathbb{R}^n$. We can then again ask the question, does the tangent space TM look like $M \times \mathbb{R}^n$ globally? We do not even need to restrict ourselves to the real case and can even look at the complex case. As it turns out, manifolds with this property, which is called being *parallelizable*, carry other properties as well, such as being orientable. Hence, knowing something about the tangent space of a manifold also tells you something about other aspects of the manifold as well.

Upon further abstraction, we arrive at the notion of a vector bundle.

Definition 1.1.1 (Vector bundle)**.** A *vector bundle* over a base space B is a map $p\colon E \to B$ with a (real or complex) vector space structure on $p^{-1}(b)$ for every $b \in B$, satisfying the following local triviality condition: there exists an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of B such that for each U_α there exists a homeomorphism $h_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ restricting to a linear isomorphism in each fiber $p^{-1}(b)$.

The space B is called the *base space*, where E is called the *total space* and the homeomorphisms h_α are called *local trivializations*. If the field $\mathbb{K} = \mathbb{R}$ we speak of *real vector bundle* and analogously if $\mathbb{K} = \mathbb{C}$ we call this a *complex vector bundle*. Often times a vector bundle $p\colon E \to B$ is abbreviated to a vector bundle E, leaving the rest implicit.

Note that the dimension of the vector spaces $p^{-1}(b)$ need not be constant, although the local trivializations do force the dimension to be *locally* constant. When the dimension $\dim(p^{-1}(b)) = n$ for all $b \in B$, we speak of a *rank-n vector bundle*.

Intuitively, a vector bundle can be thought of as attaching a vector space to every point $b \in B$ of a base space in a continuous way. The local trivializations allow us to locally examine the vector bundle as though it was simply $U_\alpha \times \mathbb{K}^n$, but globally the structure can be exotic.

Next we will discuss some examples of vector bundles, starting with:

Example 1.1.2. The *trivial bundle* given by $E = B \times \mathbb{K}^n$, with $p\colon B \times \mathbb{K}^n \to B, (b, v) \mapsto b$.

Example 1.1.3. The tangent bundle TS^2, with projection map $p\colon TS^2 \to S^2, (x, v) \mapsto x$

Example 1.1.4. More generally, the tangent bundle TM of a differentiable manifold M, where $p\colon TM \to M$ projects each element of TM to the corresponding point in M.

Example 1.1.5. The *Möbius bundle*, taking S^1 as base space and letting E be the quotient $(I \times \mathbb{R})/(0, v) \sim (1, -v)$ with projection map $p\colon E \to S^1, (x, v) \mapsto \mathrm{e}^{2\pi i x}$.

Example 1.1.6. The *tautological line bundle* over the real projective space $\mathbb{R}\mathrm{P}^n$, with as total space $E = \{(l, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \mid v \in l\}$ the space of all pairs (l, v) such that v lies in the line l, and as projection map $p\colon E \to \mathbb{R}\mathrm{P}^n, (l, v) \mapsto l$.

Since we now have a sense of what a vector bundle is, we want to know what maps between these objects are. We would like the maps to be compatible with the topological and linear structure on the vector bundle. This gives rise to the following definition.

Definition 1.1.7 (Morphism of vector bundles)**.** Given two vector bundles $p_1\colon E_1 \to B_1$ and $p_2\colon E_2 \to B_2$ a *morphism from E_1 to E_2* between these vector bundles is a pair of continuous maps $f\colon E_1 \to E_2$ and $g\colon B_1 \to B_2$ such that $g \circ p_1 = p_2 \circ f$ and that for every $b \in B_1$, the restriction $f\colon p_1^{-1}(b) \to p_2^{-1}(g(b))$ is a linear map.

Essentially, we want two maps f and g such that the following diagram commutes and f restricts to a linear map in each fiber.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

If we take a closer look at the diagram, it seems that once we have the function f, the function g is forced. This is the case, as will be discussed later in Section 1.2.

Similarly we can define an *isomorphism of vector bundles* to be a morphism of vector bundles with an inverse. This implies the base spaces and the total spaces are homeomorphic to each other and the linear structure of the vector bundle is preserved. Due to all structure being preserved, isomorphic vector bundles are often viewed the same, and we write $E_1 \cong E_2$. If there exists an isomorphism between a vector bundle $p_1\colon E \to B$ and the n-dimensional trivial bundle $p_2\colon B \times \mathbb{K}^n \to B$ for some $n \in \mathbb{N}$, the bundle E is said to be *trivializable* or is called the *trivial bundle*.

Given a base space B, one often views vector bundles up to isomorphism. Since being isomorphic is an equivalence relation, we can consider the *set of vector bundles up to isomorphism* as well, defined as

$$\mathrm{Vect}_{\mathbb{K}}(B) := \{\text{Vector bundles over } B\}/\cong .$$

We can also consider vector bundles with a fixed dimension n up to isomorphism, resulting in $\mathrm{Vect}^n_{\mathbb{K}}(B)$.

An example of a morphism between vector bundles is an embedding.

Example 1.1.8. Consider the Möbius bundle $p\colon E = (I \times \mathbb{R})/(0, v) \sim (1, -v) \to S^1$, then we can embed it into the vector bundle $\mathbb{R}^2 \times \mathbb{R}^2$ by the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & \mathbb{R}^2 \times \mathbb{R}^2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ S^1 & \xrightarrow{g} & \mathbb{R}^2 \end{array}$$

where f is given by

$$\begin{aligned} f\colon (I\times\mathbb{R})/(0,v)\sim(1,-v) &\to \mathbb{R}^2\times\mathbb{R}^2,\\ (x,v) &\mapsto \big((\cos(2\pi x),\sin(2\pi x)), v(\cos(\pi x),\sin(\pi x))\big), \end{aligned}$$

and g being forced by f to be

$$\begin{aligned} g\colon S^1 &\to \mathbb{R}^2,\\ x &\mapsto (\cos(2\pi x),\sin(2\pi x)). \end{aligned}$$

To check this is indeed a morphism we must check that f is continuous and restricts to a linear map in each fiber. The continuity of f follows from the fact that f is a composition of continuous functions and the periodicity of the sine and cosine functions. To check the linear property, fix $x\in S^1$, then f restricts to a map

$$\begin{aligned} f|_{p_1^{-1}(x)}\colon p_1^{-1}(x) &\to \{(\cos(2\pi x),\sin(2\pi x))\}\times\mathbb{R}^2\\ v &\mapsto v(\cos(\pi x),\sin(\pi x)) \end{aligned}$$

which is linear on its image spanned by the vector $(\cos(\pi x),\sin(\pi x))\in\mathbb{R}^2$.

Up until here our point of view has been to attach a vector space to each point of a base space B, and then giving an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of B such that local trivializations $h_\alpha\colon p^{-1}(U_\alpha)\to U_\alpha\times\mathbb{K}^n$ exist. We can also reverse the reasoning and start with an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of B, and then patch together different $U_\alpha\times\mathbb{K}^n$ to form a total space E. The advantage of this point of view is that we only need to know the open cover, and how to glue them together in a consistent manner. Considering this, we construct a vector bundle as follows:

Lemma 1.1.9. *Given a base space B, an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of B and a family functions $\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_n(\mathbb{K})\}$ satisfying the following conditions*

1. $g_{\alpha\alpha}=\mathrm{Id}$ *on* U_α,
2. $g_{\alpha\beta}g_{\beta\alpha}=\mathrm{Id}$ *on* $U_\alpha\cap U_\beta$,
3. $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=\mathrm{Id}$ *on* $U_\alpha\cap U_\beta\cap U_\gamma$.

where the multiplication is pointwise matrix multiplication in $GL_n(\mathbb{K})$, then the space

$$E=\left(\bigsqcup_{\alpha\in\mathcal{A}}\{\alpha\}\times U_\alpha\times\mathbb{K}^n\right)\Big/\sim$$

with equivalence relation $\sim$ given by

$$(\beta,b,v)\sim(\alpha,b,g_{\alpha\beta}(b)v)$$

for all $\alpha,\beta\in\mathcal{A}, b\in U_\alpha\cap U_\beta, v\in\mathbb{K}^n$

with projection map $p\colon E\to B,(\alpha,b,v)\mapsto b$ is a vector bundle.

Proof. Let us begin by checking that $\sim$ is indeed an equivalence relation. The relation is reflexive since

$$(\alpha,b,v)\sim(\alpha,b,g_{\alpha\alpha}(b)v)=(\alpha,b,\mathrm{Id}\,v)=(\alpha,b,v).$$

It is also symmetric since if $(\beta,b,v)\sim(\alpha,b,g_{\alpha\beta}(b)v)$ then it follows

$$(\alpha,b,g_{\alpha\beta}(b)v)\sim(\beta,b,g_{\beta\alpha}(b)g_{\alpha\beta}(b)v)=(\beta,b,\mathrm{Id}\,v)=(\beta,b,v)\,.$$

Finally, for transitivity, if

$$(\gamma,b,v)\sim(\beta,b,g_{\beta\gamma}(b)v)\text{ and }(\beta,b,g_{\beta\gamma}(b)v)\sim(\alpha,b,g_{\alpha\beta}(b)g_{\beta\gamma}(b)v),$$

then

$$(\alpha, b, g_{\alpha\beta}(b)g_{\beta\gamma}(b)v) \sim (\gamma, b, g_{\gamma\alpha}(b)g_{\alpha\beta}(b)g_{\beta\gamma}(b)v) = (\gamma, b, \mathrm{Id}\, v) = (\gamma, b, v).$$

For E to be a vector bundle, we must check that p is continuous, $p^{-1}(b)$ has a vector space structure for every $b \in B$ and that the local triviality condition is satisfied. First, let $W \subseteq B$ be open. Then $p^{-1}(W)$ is open in E as well by the properties of the product and quotient topology, which makes p continuous. Second, given $b \in B$, we can find an open U_α containing b. Then $p^{-1}(b) \subseteq p^{-1}(U_\alpha) \cong \{\alpha\} \times U_\alpha \times \mathbb{K}^n$ and by this we see $p^{-1}(b) \cong \{\alpha\} \times \{p\} \times \mathbb{K}^n$ which endows it with a vector space structure. This structure is independent of the choice of open U_α, since if U_β is another open, $g_{\alpha\beta}$ restricts to a linear isomorphism between these two vector space structures. Third, the local trivializations are precisely $p^{-1}(U_\alpha) \cong \{\alpha\} \times U_\alpha \times \mathbb{K}^n$ by construction. □

The family of functions $\{g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{K})\}$ are often called *gluing functions* and the conditions imposed on them is te ensure the gluing functions are compatible with each other.

The next natural question to ask is given a vector bundle E_1, can we find gluing functions such that the vector bundle E_2 constructed with these functions in the manner of Lemma 1.1.9 is isomorphic to E_1? The answer is yes, and the key observation is the following. Consider a vector bundle E and local trivializations $h_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$. Take a point $b \in U_\alpha \cap U_\beta$, then the composition $h_{\alpha\beta} := h_\alpha \circ h_\beta^{-1} \colon \{b\} \times \mathbb{K}^n \to \{b\} \times \mathbb{K}^n$ is the composition of linear isomorphisms and hence corresponds to a unique element in $GL_n(\mathbb{K})$. Following this line of reasoning, we find an element in $GL_n(\mathbb{K})$ for every $b \in U_\alpha \cap U_\beta$ and we can define a function

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{K}),$$

such that $h_{\alpha\beta}(b, v) = (b, g_{\alpha\beta}(b)v)$. The function $g_{\alpha\beta}$ will be continuous by continuity of $h_{\alpha\beta}$. We also see the functions $g_{\alpha\beta}$ satisfy the desired properties stated in Lemma 1.1.9, which can be checked by examining the corresponding functions $h_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}$. Motivated by the idea of viewing a vector bundle in terms of its gluing functions we state the following result:

Proposition 1.1.10. *Given a vector bundle $p_1 \colon E_1 \to B$ with local trivializations $h_{1\alpha} \colon p_1^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$, then the vector bundle E_2 constructed using the gluing functions $g_{2\alpha\beta}$ induced by the compositions $h_{1\alpha} \circ h_{1\beta}^{-1}$ is isomorphic to E_1.*

Proof. To prove this statement, we must ensure we can find a homeomorphism $f \colon E_1 \to E_2$ restricting to linear isomorphism in each fiber. This function f is given by

$$\begin{aligned} f \colon E_1 &\to E_2 \\ (b, v) &\mapsto [\alpha, h_\alpha(b, v)], \end{aligned}$$

where α is chosen such that $b \in U_\alpha$. To see f is well-defined, consider $b \in U_\alpha \cap U_\beta$, then $[\beta, h_\beta(b, v)] = [\alpha, h_{\alpha\beta} h_\beta(b, v)] = [\alpha, h_\alpha(b, v)]$. We also need to check f is a homeomorphism. To check f is continuous, consider the composition

$$U_\alpha \times \mathbb{K}^n \xrightarrow{h_{2\alpha}^{-1}} p_2^{-1}(U_\alpha) \xrightarrow{f} p_1^{-1}(U_\alpha) \xrightarrow{h_{2\alpha}} U_\alpha \times \mathbb{K}^n$$

which is the identity and hence continuous. Since $h_{i\alpha}$ are homeomorphisms, it follows f must be continuous. We construct an inverse in the only way possible, namely

$$\begin{aligned} f^{-1} \colon E_2 &\to E_1 \\ [\alpha, b, v] &\mapsto (b, h_\alpha^{-1}(v)) \end{aligned}$$

By a similar check as before, we see f^{-1} is well-defined and continuous. The last thing to check is the functions restrict to linear isomorphism on each fiber. Fix $b \in B$, then

$$f|_{p^{-1}(b)} \colon (b, v) \mapsto [\alpha, h_\alpha(b, v)] \text{ if } b \in U_\alpha$$

is a linear isomorphism since h_α restricts to a linear isomorphism on $p^{-1}(b)$. □

This proposition shows it does not matter if we view vector bundles as attaching a vector space to each point of a base space or as a patching of $U_\alpha \times \mathbb{K}^n$, and it shows explicitly how to move between the two points of view. This will prove to be helpful, as both perspectives provide us with interesting examples and proof techniques.

1.2 Sections

Now that we have a bit of understanding of vector bundles, it would be nice to develop some tools to better handle these objects. The first of such tools will be a *section*, defined as follows:

Definition 1.2.1 (Section)**.** A *section* of a vector bundle $p\colon E \to B$ is a continuous map $s\colon B \to E$ sending each point $b \in B$ to a vector $s(b) \in p^{-1}(b)$.

An alternative way of phrasing the condition $s(b) \in p^{-1}(b)$ is to say $p \circ s = \mathrm{Id}$. As an example, every section of the tangent space of a manifold gives a vector field on the manifold.

Every vector bundle admits one canonical section, the *zero section*, which is defined as $s\colon B \to E, b \mapsto (b, 0)$. It is customary to identify the base space B with the zero section, since s and p restricted to the zero section define a homeomorphism between the two. This also gives some insight as to why, when considering morphisms of vector bundles, the map $f\colon E_1 \to E_2$ determines the map $g\colon B_1 \to B_2$. The map g can be seen as f restricted to the zero section.

Sections can be used to distinguish vector bundles, as can be seen in the following examples:

Example 1.2.2. We want to show the Möbius bundle is not isomorphic to the trivial bundle. Looking at the complement of the zero section of both bundles, the complement of the zero section of the Möbius bundle is connected, while this is not true for the trivial bundle, and hence they can not be isomorphic since any isomorphism is also a homeomorphism.

Example 1.2.3. Consider the tautological line bundle

$$p_1\colon E = \{(l, v) \in \mathbb{R}\mathrm{P}^n \times \mathbb{R}^{n+1} \mid v \in l\} \to \mathbb{R}\mathrm{P}^n,$$

for $n \geq 1$. We wish to distinguish it from the one dimensional trivial bundle $p_2\colon \mathbb{R}\mathrm{P}^n \times \mathbb{R} \to \mathbb{R}\mathrm{P}^n$. Both bundles are one-dimensional bundles over the same base space, but the trivial bundle admits a section which is nowhere zero, while the tautological line bundle does not. To see the tautological line bundle does not admit a nowhere zero section, let

$$\begin{aligned} s\colon \mathbb{R}\mathrm{P}^n &\to E \\ x &\mapsto (x, \tilde{s}(x)) \end{aligned}$$

be such a section. Considering the quotient map $q\colon S^n \to \mathbb{R}\mathrm{P}^n$, we can use q to define a function

$$\begin{aligned} t\colon S^n &\to \mathbb{R}\mathrm{P}^n \to E \\ x &\mapsto (q(x), \tilde{s}(q(x))). \end{aligned}$$

The function t is the composition of continuous functions and hence continuous. In each fiber $p_1^{-1}(b)$, the point $(q(x), \tilde{s}(q(x)))$ can be written as $(q(x), \tilde{t}(x)x)$ since the fiber $p_1^{-1}(b)$ consists of scalar multiples of $q(x)$. The induced function $\tilde{t}$ is continuous by continuity of t. The function $\tilde{t}\colon S^n \to \mathbb{R}$ has the property $\tilde{t}(-x) = -\tilde{t}(x)$. This can be seen by noting $t(x) = t(-x)$ and considering

$$\begin{aligned} t(x) &= (q(x), \tilde{t}(x)x) \\ t(-x) &= (q(-x), \tilde{t}(-x)(-x)) = (q(x), -\tilde{t}(-x)x), \end{aligned}$$

and the property follows by comparing the last components. If we fix $x_0 \in S^n$, then by S^n being path-connected for $n \geq 1$ we can find a path $\gamma\colon I \to S^n$ with $\gamma(0) = x_0$ and $\gamma(1) = -x_0$. Composing the path with $\tilde{t}$, we get a function

$$\tilde{t} \circ \gamma\colon I \to E,$$

with $-\tilde{t} \circ \gamma(0) = -\tilde{t}(x_0) = \tilde{t}(-x_0) = \tilde{t} \circ \gamma(1)$. By the intermediate value theorem, there must exist a $c \in I$ such that $\tilde{t} \circ \gamma(c) = 0$. If we set $\gamma(c) = x_1$ and compute $t \circ \gamma(c) = t(x_1)$, we see

$$t(x_1) = (q(x_1), \tilde{s}(q(x_1))) = (q(x_1), \tilde{t}(x_1)x_1) = (q(x_1), \tilde{t}(\gamma(c))x_1) = (q(x_1), 0)$$

which in turn implies the section s hits zero. This implies the vector bundles can not be isomorphic, since if they were we could use the isomorphism to send the nowhere vanishing section of the trivial bundle to a nowhere vanishing section of the tautological line bundle.

Further exploring the use of sections, we examine an n-dimensional vector bundle $p\colon E \to B$ with n *linearly independent sections*. This means there exist n sections $s_1, \dots, s_n$ such that the vectors $s_1(b), \dots, s_n(b)$ are linearly independent in each fiber $p^{-1}(b)$. We claim any vector bundle admitting such sections is isomorphic to the trivial bundle. To see this, consider the map

$$\begin{aligned} f\colon B \times \mathbb{K}^n &\to E \\ (b, v_1, v_2, \dots, v_n) &\mapsto \left(b, \sum_{i=1}^{n} v_i s_i(b)\right), \end{aligned}$$

which is a continuous bijection and a linear isomorphism on each fiber. It does not exactly meet the requirements of a vector bundle isomorphism since we do not know anything about its inverse, but f in fact is an isomorphism of vector bundles by the following result:

Lemma 1.2.4. *A continuous map $f\colon E_1 \to E_2$ between vector bundles over the same base space B is an isomorphism if and only if $p_2 \circ f = p_1$, and for every $b \in B$ the induced map $p_1^{-1}(b) \to p_2^{-1}(b)$ is linear isomorphism.*

Proof. Clearly if f is an isomorphism of vector bundles it satisfies the conditions stated in the lemma. Conversely, for f to be an isomorphism of vector bundles it must be a homeomorphism between E_1 and E_2 and restrict to a linear isomorphism in each fiber.

Since f takes each fiber to each corresponding fiber by linear isomorphism, it must be a continuous bijection. It also already satisfies the condition of restricting to linear isomorphism on each fiber. The only thing left to check is that the inverse f^{-1} is continuous. To do so, we fix an arbitrary point $b_0 \in B$ and check f^{-1} is continuous at b_0. First, find neighborhoods U_α and U_β containing b_0 over which E_1 and E_2 respectively are trivial. Setting $U := U_\alpha \cap U_\beta$, both bundles are trivial over U. Now compose f with the local trivializations $h_\alpha\colon E_1 \supseteq p_1^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ and $h_\beta\colon E_2 \supseteq p_2^{-1}(U_\beta) \to U_\beta \times \mathbb{K}^n$ to obtain

$$\tilde{f} := h_\beta \circ f \circ h_\alpha\colon U \times \mathbb{K}^n \to U \times \mathbb{K}^n.$$

Since $\tilde{f}$ is a composition of functions which restrict to linear isomorphisms, we deduce $\tilde{f}$ is function of the form $\tilde{f}(b, v) = (b, g(b)v)$ with $g\colon U \to GL_n(\mathbb{K})$ a continuous function. Regarding $g(b)$ as a matrix, its entries depend continuously on b and so the entries of the inverse matrix $g^{-1}(b)$ do as well, since the inverse mapping is continuous. This yields $\tilde{f}^{-1}\colon (b, v) \to (b, g(b)^{-1}v)$ is a continuous function as well, and we conclude

$$h_\alpha^{-1} \circ f^{-1} \circ h_\beta^{-1} = \tilde{f}^{-1}$$

is continuous and hence f^{-1} is continuous as well at b_0. Since this holds for an arbitrary point $b_0 \in B$ we conclude f^{-1} is continuous and that f is an isomorphism of vector bundles. □

An application of the above is:

Example 1.2.5. The tangent bundle TS^3 of $S^3 \subset \mathbb{R}^4$ is trivial. This can be seen be defining sections $s_i(x) = (x, \tilde{s}_i(x))$ with

$$\begin{aligned} \tilde{s}_1(x) &= (-x_2, x_1, -x_4, x_3) \\ \tilde{s}_2(x) &= (-x_3, x_4, x_1, -x_2) \\ \tilde{s}_3(x) &= (-x_4, x_3, x_2, x_1). \end{aligned}$$

The sections s_i are motivated by the formulas for quaternion multiplication, and produce linearly independent sections. Something similar can be done for the spheres S^1 and S^7, using complex multiplication and octonion multiplication, yielding linearly independent sections on S^1 and S^7.

1.3 Inner products

When studying an object, it is always nice to endow it with extra structure. Since vector bundles are created by attaching a vector space to every point of a base space, it would be nice if we could endow a vector bundle with the some of the same structure we would a vector space. A vector bundle already has a topology and a linear structure, but we could try to lift another very important structure from vector spaces to vector bundles, the inner product. The natural way to define an inner product on a vector bundle is to require it restricts to an inner product in each fiber. The only issue is we want a function that takes in multiple elements of E, but not all pairs of elements of E are suitable to compute an inner product of. Only elements of E belonging to same fiber seem suitable. A solution is to define a new object using the *direct sum* of vector spaces. More details will be given in Section 1.4, but for now the following definition suffices:

Definition 1.3.1 (Direct sum of a vector bundles)**.** Given vector bundles E_1 and E_2 over the same base space B, the *direct sum of the vector bundles* $E_1 \oplus E_2$ is defined as

$$E_1 \oplus E_2 = \{\big((b, v), (b', v')\big) \in E_1 \times E_2 \mid b = b'\}.$$

It turns out $E_1 \oplus E_2$ is indeed a vector bundle, but this will be discussed in Section 1.4. For now, we will only consider $E \oplus E$, the direct sum of a vector bundle with itself. This object interacts nicely with our idea of an inner product, since all elements are pairs of vectors in the same fiber. Having set this up, we can give following definition:

Definition 1.3.2 (Inner product)**.** An *inner product* on a vector bundle is a map $\langle \cdot, \cdot \rangle \colon E \oplus E \to \mathbb{K}$ which restricts to an inner product on each fiber $p^{-1}(b)$.

The next natural question is, do all vector bundles admit an inner product? It turns out that certain conditions need to be imposed only on the base space B, and this is summarized in the following proposition:

Lemma 1.3.3. *Any vector bundle $p \colon E \to B$ with compact base space B can be endowed with an inner product.*

Remark 1.3.4. The condition that B must be compact can be weakened to B being paracompact.

Proof. To construct an inner product, we lift each inner product in the local trivializations and then use a partition of unity to go from a local inner product to a global one. Start with an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of B over which B is trivial. By compactness, this can be reduced to a finite subcover $\{U_i\}_{i \in \mathcal{I}}$. Now considering the local trivializations $h_i \colon p^{-1}(U_i) \to U_i \times \mathbb{K}^n$, we can pull back the standard inner product on $\mathbb{K}^n$ to $p^{-1}(U_i)$ by setting

$$\langle \cdot, \cdot \rangle_i \colon = \langle h_i(\cdot), h_i(\cdot) \rangle \colon p^{-1}(U_i) \oplus p^{-1}(U_i) \to \mathbb{K}.$$

By *Urysohn's lemma* A we can also find a partition of unity $\{\eta_i\}_{i \in \mathcal{I}}$ subordinate to $\{U_i\}_{i \in \mathcal{I}}$, since B is compact. Finally, we define the inner product on E by setting

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon E \oplus E &\to \mathbb{K} \\ \big((b, v), (b, w)\big) &\mapsto \sum_{i \in \mathcal{I}} \eta_i(b) \langle v, w \rangle_i. \end{aligned}$$

The inner product defined above restricts to an inner product on each fiber, since scaled sums of inner products yields an inner product. □

Having the inner product, we are curious to see what other ideas from vector spaces we can lift to vector bundles. One of the first things we encounter is the idea of a vector subspace and its orthogonal complement. This can be lifted to vector bundles quite painlessly in the following manner:

Definition 1.3.5 (Vector subbundle)**.** Given a vector bundle $p \colon E \to B$, a *vector subbundle* is a subspace $F \subseteq E$ intersecting each fiber of E in a vector subspace, such that the restriction $p \colon F \to B$ is a vector bundle.

Taking more inspiration from linear algebra, we state:

Proposition 1.3.6. *If $p\colon E \to B$ is a vector bundle over a compact base space B, and $F \subseteq E$ is a vector subbundle, then there exists a vector subbundle $F^\perp \subseteq E$ such that $F \oplus F^\perp \cong E$.*

Proof. The idea is to define $F^\perp$ in the obvious way, and then showing this is a vector subbundle. Since B is a compact base space, we can endow E with an inner product. Then we can define $F^\perp$ as the orthogonal complement of F in each fiber

$$F^\perp := \bigsqcup_{b \in B} \left(p|_F^{-1}(b)\right)^\perp .$$

We claim that together with the natural projection map $p\colon F^\perp \to B$, this is a vector subbundle. If this is the case, then we can define a map

$$\begin{aligned} f\colon F \oplus F^\perp &\to E \\ \big((b,v),(b,w)\big) &\mapsto (b, v+w) \end{aligned}$$

which will be an isomorphism due to Lemma 1.2.4.

For $F^\perp$ to be a vector bundle, it must satisfy the local triviality condition. To see that it does, fix $b_0 \in B$, then we can find opens $b_0 \in U_\alpha$ and $b_0 \in U_\beta$ such that E and F are trivial over U_α and U_β respectively. We set $U := U_\alpha \cap U_\beta$. If F is an m-dimensional bundle, we can find m locally independent sections using the trivialization $h_\beta\colon p^{-1}(U) \to U \times \mathbb{K}^m$ by

$$\begin{aligned} s_i\colon U &\to p^{-1}(U) \\ b &\mapsto h_\beta^{-1}(b, e_i) \end{aligned}$$

where e_i is the i-th standard basis vector in $\mathbb{K}^m$. We can enlarge this set of m independent sections to a total of n independent sections of E by choosing $s_{m+1}, \dots, s_n$ first in the fiber $p^{-1}(b_0)$ using the local trivialization h_α, and then extending to U. The sections $s_1, \dots, s_m, s_{m+1}, \dots, s_n$ will remain independent in a small neighborhood $W \subseteq U$ of b_0 by continuity of the determinant function. Now we can apply the Gram-Schmidt orthogonalization process D with respect to the inner product on E to all sections $s_1, \dots, s_n$ in each fiber $p^{-1}(b)$ for $b \in W$ to obtain new sections $s'_1, \dots, s'_n$, which are all still continuous maps since the Gram-Schmidt process is continuous. We can use the sections $s'_1, \dots, s'_n$ to define a local trivialization of E over W by defining

$$\begin{aligned} h\colon p^{-1}(W) &\to W \times \mathbb{K}^n \\ s'_i(b) &\mapsto (b, e_i). \end{aligned}$$

This trivialization carries F to $W \times \mathbb{K}^m$ and $F^\perp$ to $W \times \mathbb{K}^{n-m}$, so h trivializes $F^\perp$. To obtain a cover of B over which $F^\perp$ trivializes, all we need to do is repeat this process for all $b \in B$ to obtain corresponding opens W_α with local trivializations. Lastly, we need to check $f\colon F \oplus F^\perp \to E$ is an isomorphism. The map f is continuous since it is defined using only addition. When restricted to a fiber $p^{-1}(b)$, f is a linear isomorphism since in each fiber, $F^\perp$ is the orthogonal complement of F. Lemma 1.2.4 then ensures $F \oplus F^\perp \cong E$. □

If we set $F = E$, the last part of the proof shows that the local trivializations of any vector bundle with an inner product can be chosen to carry the inner product of E to the standard inner product on $\mathbb{K}^n$, and so these local trivializations are isometries.

1.4 Operations on vector bundles

In Section 1.3 we have already seen a way to make a new vector bundle out of existing ones using the direct sum. In this section we will explore more ways of doing so.

Perhaps one of the first operations one wishes to explore is the Cartesian product. It is defined as follows:

Definition 1.4.1 (Cartesian product of vector bundles)**.** Let $p_1 \colon E_1 \to B_1$ and $p_2 \colon E_2 \to B_2$ be two vector bundles, then the *Cartesian product of* E_1 *and* E_2 is defined as

$$p_1 \times p_2 \colon E_1 \times E_2 \to B_1 \times B_2.$$

Lemma 1.4.2. *The Cartesian product of two vector bundles is again a vector bundle.*

Proof. The fibers of $E_1 \times E_2$ are given by $p_1^{-1}(b_1) \times p_2^{-1}(b_2)$, which are again vector spaces, and an open cover over which $E_1 \times E_2$ is trivial is given by $\{U_\alpha \times U_\beta\}_{(\alpha,\beta)\in\mathcal{A}\times\mathcal{B}}$, where $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{U_\beta\}_{\beta\in\mathcal{B}}$ are covers over which E_1 and E_2 are trivial respectively. Locally E_1 is n-dimensional and E_2 is m-dimensional, with local trivializations are then given by

$$h_\alpha \times h_\beta \colon p_1^{-1}(U_\alpha) \times p_2^{-1}(U_\beta) \to U_\alpha \times U_\beta \times \mathbb{R}^{n+m},$$

which shows $E_1 \times E_2$ is an $n+m$-dimensional vector bundle. □

Next we take a general idea in mathematics, the pullback by a function.

Definition 1.4.3 (Pullback bundle)**.** Given a vector bundle $p \colon E \to B$ and a continuous map $g \colon A \to B$, the *pullback of* E *by* g is defined as $g^*(E) := \{\big(a,(b,v)\big) \in A \times E \mid g(a) = b\}$.

The definition can be summarized as the vector bundle that makes the following diagram commute:

$$\begin{array}{ccc} g^*(E) & \xrightarrow{f} & E \\ \big\downarrow{\scriptstyle p^*} & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & B \end{array}$$

where $f(a,(b,v)) = (b,v)$.

Lemma 1.4.4. *The pullback bundle* $p^* \colon g^*(E) \to A$ *is a vector bundle*

Proof. First we must check every fiber $(p^*)^{-1}(a)$ has a vector space structure. The set

$$(p^*)^{-1}(a) = \{\big(a,(b,v)\big) \in \{a\} \times E \mid g(a) = b\}$$

carries a natural vector space structure from E by $\big(a,(b,v)\big) + \big(a,(b,w)\big) = \big(a,(b,v+w)\big)$ and $\lambda\big(a,(b,v)\big) = \big(a,(b,\lambda v)\big)$. For the local triviality, let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of B over which B trivializes, then an open cover of A is given by $\{W_\alpha\}_{\alpha\in\mathcal{A}} := \{g^{-1}(U_\alpha)\}_{\alpha\in\mathcal{A}}$. We can find local trivializations $h_\alpha^* \colon (p^*)^{-1}(W_\alpha) \to W_\alpha \times \mathbb{K}^n$ by setting

$$h_\alpha^* \colon \big(a,(b,v)\big) \mapsto \big(a, h_\alpha(b,v)\big).$$

This shows the pullback bundle $g^*(E)$ is a vector bundle. □

Some properties of the pullbacks are given in the following lemma:

Lemma 1.4.5. *Given a vector bundle* $p \colon E \to C$ *and maps* $g_1 \colon A \to B$, $g_2 \colon B \to C$, *then*

1. $g_2^* \circ g_1^*(E) \cong (g_1 \circ g_2)^*(E)$
2. $\mathrm{Id}^*(E) \cong E$

Proof. For the first claim, consider the isomorphism

$$\begin{aligned} f \colon g_2^* \circ g_1^*(E) &\to (g_1 \circ g_2)^*(E) \\ (a,(b,(c,v))) &\mapsto (a,(c,v)) \end{aligned}$$

The function f is continuous since it only drops a coordinate and its inverse is given by

$$\begin{aligned} f^{-1} \colon (g_1 \circ g_2)^*(E) &\to g_2^* \circ g_1^*(E) \\ (a,(c,v)) &\mapsto (a,(g(a),(c,v))) \end{aligned}$$

The inverse of f is a composition of continuous functions, and hence f is a homeomorphism. Each fiber is also linearly mapped onto each corresponding fiber by f, and by Lemma 1.2.4 we conclude f is an isomorphism.

For the second claim, we define the function

$$\begin{aligned} f\colon \operatorname{Id}^*(E) &\to E \\ (b, (\operatorname{Id}(b), v)) &\mapsto (b, v), \end{aligned}$$

which is a homeomorphism and a linear mapping on each fiber and hence by Lemma 1.2.4 is an isomorphism. □

Example 1.4.6. An example of the pullback bundle is the *restriction bundle*. Given a vector bundle $p\colon E \to B$ and a subset $A \subseteq B$, we can restrict E to A by pulling back E using the inclusion $i\colon A \hookrightarrow B$. This gives a vector bundle

$$i^*(E) := \{\big(a, (b, v)\big) \in A \times E \mid i(a) = b\} \to A,$$

which is precisely E restricted to A, denoted as $E|_A$.

Revisiting the direct sum of two vector bundles, we observe we took an existing operation on vector spaces, the direct sum, and applied it fiberwise to create a new vector bundle. The direct sum is not the only operation on vector spaces, there are many more such as the *tensor product* B and the *k-th exterior power*. One can also consider the vector space $\operatorname{Hom}(V, W)$ of all linear maps from V to W, or the dual vector space $\operatorname{Hom}(V, \mathbb{K})$. All these operations can be applied to vector bundles fiberwise, and we would like to know if the resulting object is again a vector bundle. To answer this question, we must first formalize what we mean by an operation on vector spaces, which we will do using category theory [Mac71].

Definition 1.4.7. Let $\mathfrak{V}$ be the category of finite dimensional vector spaces and all isomorphisms between them. A *functor (in two variables)* $F\colon \mathfrak{V} \times \mathfrak{V} \to \mathfrak{V}$ is an operation which assigns

1. to each pair $V, W \in \mathfrak{V}$ a vector space $F(V, W) \in \mathfrak{V}$;
2. to each pair of isomorphisms $f\colon V_1 \to V_2$, $g\colon W_1 \to W_2$ an isomorphism $F(f, g)\colon F(V_1, W_1) \to F(V_2, W_2)$;
3. with $F(\operatorname{Id}_V, \operatorname{Id}_W) = \operatorname{Id}_{F(V,W)}$;
4. and $F(f_2 \circ f_1, g_2 \circ g_1) = F(f_1, g_1) \circ F(f_2, g_2)$.

A *functor of k variables* is defined in a similar fashion.

Some examples of functors of this kind are exactly the operations discussed above, namely the direct sum, the tensor product, the k-th exterior power, the space $\operatorname{Hom}(V, W)$ and the dual vector space.

Having the idea of a functor on vector spaces in k variables, we can state the following theorem:

Theorem 1.4.8. *Let $F\colon \mathfrak{V} \times \cdots \times \mathfrak{V} \to \mathfrak{V}$ be a functor of k variables and let $E_1, \ldots E_k$ be k vector bundles over the same base space B. Then the disjoint union*

$$E := \bigsqcup_{b \in B} F(p_1^{-1}(b), \ldots, p_k^{-1}(b))$$

together with the projection map $p\colon E \to B$ can be equipped with a topology such that E is a vector bundle. The bundle will be denoted by $F(E_1, \ldots, E_k)$.

Proof. We want to equip E with a topology such that the projection map $p\colon E \to B$ is continuous and such that we can find local trivializations covering E. The way this is done is by reverse engineering the topology on E. We start with maps that will later become the local trivializations, and then choose a topology such that these maps will become continuous.

Each bundle E_i has an open cover over which it trivializes. Intersecting all open covers, we obtain an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of B over which all vector bundles trivialize. For each $b \in B$, we define

$$\begin{aligned} h_{ib} \colon p_i^{-1}(b) &\to \mathbb{R}^{n_i} \\ (b, v) &\mapsto h_i(v), \end{aligned}$$

where by $h_i(v)$ we mean the restriction of $h_i \colon p_i^{-1}(b) \to \{b\} \times \mathbb{R}^n$ to its last component. The map h_{ib} is a linear isomorphism, and we can define a map which again will be a linear isomorphism by the property of functors:

$$F(h_{1b}, \ldots, h_{kb}) \colon F(p_1^{-1}(b), \ldots, p_k^{-1}(b)) = p^{-1}(b) \to F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$$

Using this isomorphism, we can define what will become local trivialization for an open U_α as

$$\begin{aligned} h_\alpha \colon p^{-1}(U_\alpha) &\to U_\alpha \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \\ (b, v) &\mapsto (b, F(h_{1b}, \ldots, h_{kb})(b, v)). \end{aligned}$$

By construction, h_α is a bijection and restricts to a linear isomorphism on each fiber, but it is not yet continuous, nor does it have a continuous inverse. But, since we can choose our topology on E, we will choose it in such a way the trivializations will become continuous. The topology on E is defined as

$$U \subseteq E \text{ is open if and only if } h_\alpha(U) \subseteq U_\alpha \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \text{ is open for some } \alpha \in \mathcal{A}.$$

This ensures both h_α and h_α^{-1} are continuous. The only thing left to check is the topology agrees on intersections $p^{-1}(U_\alpha \cap U_\beta)$. To see this is the case, consider the following diagram:

$$\begin{array}{ccc} p^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{\quad h_\alpha \quad} & (U_\alpha \cap U_\beta) \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \\ \downarrow{\scriptstyle h_\beta} & & \downarrow{\scriptstyle h_\beta^{-1}} \\ (U_\alpha \cap U_\beta) \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) & \xrightarrow{\quad h_\alpha^{-1} \quad} & p^{-1}(U_\alpha \cap U_\beta) \end{array}$$

which shows that the topology on E is forced to be compatible with both h_α and h_β. Given an open $W \subseteq p^{-1}(U_\alpha \cap U_\beta)$, then by definition $h_\alpha(W) \subseteq (U_\alpha \cap U_\beta) \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$ is open as well. But then $h_\beta^{-1}(h_\alpha(W)) \subseteq p^{-1}(U_\alpha \cap U_\beta)$ is also open. Hence, on any subset which is open in the topology induced by h_α is also open in the topology induced by h_β. Lastly we need to check $p \colon E \to B$ is continuous, which can be seen by letting $U \subseteq B$ be open and considering $h_\alpha(p^{-1}(U)) = U \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$ which is open. This shows E can be equipped with a topology such that it becomes a vector bundle. □

Let us revisit the direct sum. Theorem 1.4.8 ensures the direct sum bundle $E_1 \oplus E_2$ is indeed a vector bundle. Having this assurance, let us take a look at some examples:

Example 1.4.9. The direct sum of two trivial bundles $B \times \mathbb{K}^n \oplus B \times \mathbb{K}^m \cong B \times \mathbb{K}^{n+m}$ which is again a trivial bundle.

Example 1.4.10. The direct sum of the tangent bundle TS^n and the *normal bundle* $p \colon NS^n \to S^n$ with

$$NS^n := \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid v = \lambda x \text{ for some } \lambda \in \mathbb{R}\},$$

where we view S^n embedded inside $\mathbb{R}^{n+1}$ as all points with unit distance from the origin. If we take the direct sum $TS^n \oplus NS^n$ we obtain

$$TS^n \oplus NS^n = \{\big((x, v), (x', v')\big) \in TS^n \times NS^n \mid x = x'\}.$$

Summing the tangent bundle with the normal bundle, we might expect to obtain the trivial bundle $S^n \times \mathbb{R}^{n+1}$, which is the case by the isomorphism

$$\begin{aligned} f \colon TS^n \oplus NS^n &\to S^n \times \mathbb{R}^{n+1} \\ (x, v, tx) &\mapsto (x, v + tx) \end{aligned}$$

and we see we can sum two non-trivial bundles to the trivial bundle.

Example 1.4.11. Taking the base space $S^0 \cong \{x, y\}$ the set with two elements and the discrete topology, consider the vector bundles $E_1 = \{x\} \times \mathbb{K} \cup \{y\} \times \mathbb{K}^n$ and $E_2 = \{x\} \times \mathbb{K} \cup \{y\} \times \mathbb{K}^m$. The direct sum $E_1 \oplus E_2$ is the bundle

$$E_1 \oplus E_2 = \{x\} \times \mathbb{K}^2 \cup \{y\} \times \mathbb{K}^{n+m}$$

which is not trivial.

Example 1.4.12. We can also consider the direct sum of the Möbius bundle with itself. This turns out to be the trivial bundle, which can be seen geometrically by taking the Möbius bundle as being embedded in $S^1 \times \mathbb{R}^2$, and then considering the orthogonal complement of the Möbius bundle. In each fiber $\{x\} \times \mathbb{R}^2$, taking the orthogonal complement of the Möbius bundle will yield the line in $\{x\} \times \mathbb{R}^2$ making an angle of 90 degrees with the fiber of the Möbius bundle. The union of these orthogonal complements will form a second copy of the Möbius bundle, which results in the decomposition of the trivial bundle as the sum of two Möbius bundles.

Above there are several examples where the sum of two non-trivial bundles results in a trivial bundle. It turns out there is a general result:

Proposition 1.4.13. *Given a vector bundle E_1 over a compact base space B, there exists a bundle E_2 such that $E_1 \oplus E_2$ is trivial*

Proof. The idea of the proof is to find an embedding $E_1 \hookrightarrow B \times \mathbb{K}^N$ and then use Proposition 1.3.6 to find its orthogonal complement. First, we start by constructing a suitable open cover. Let $b \in B$, then there exists an open $b \in U_b$ such that E is trivial over U_b. By *Urysohn's lemma* A there exists a map $\eta_b \colon B \to [0, 1]$ which is zero outside of U_b and non-zero at b. Repeating this process for every b, we get a family of maps $\{\eta_b\}_{b \in B}$ and we can construct an open cover $\{\eta_b^{-1}(0, 1]\}_{b \in B}$ of B. By compactness of B, this open cover can be reduced to a finite open cover, which we will relabel to $\{\eta_i^{-1}(0, 1]\}_{i \in \mathcal{I}}$ with corresponding opens U_i. Next, we define functions $\tilde{f}_i \colon E \to \mathbb{R}^n$ by

$$\tilde{f}_i(b, v) = \eta_i(b) h_i(v),$$

where by $h_i(v)$ we mean the restriction of $h_i \colon p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ to its last component. The functions $\tilde{f}_i$ are the rescaling of linear isomorphisms and hence are linear injections on each fiber over $\eta_i^{-1}(0, 1]$. We now define a function $\tilde{f}$ with coordinates $\tilde{f}_i$ by

$$\begin{aligned} \tilde{f} \colon E &\to \mathbb{R}^N \\ (b, v) &\mapsto \big(\tilde{f}_1(b, v), \ldots, \tilde{f}_m(b, v)\big). \end{aligned}$$

Since all $\tilde{f}_i$ are linear injections on each fiber, it follows $\tilde{f}$ is also a linear injection on each fiber of E_1. Finally, we define our embedding

$$\begin{aligned} f \colon E &\hookrightarrow B \times \mathbb{R}^N \\ (b, v) &\mapsto (b, \tilde{f}(b, v)) \end{aligned}$$

The image of f is a vector bundle, with local trivialization $h_i \colon p^{-1}\big(\eta_i^{-1}(0, 1]\big) \to \eta_i^{-1}(0, 1] \times \mathbb{R}^n$ given by restricting to the i-th $\mathbb{R}^n$ factor of $\mathbb{R}^N$. We conclude we can view E_1 as a subbundle of $B \times \mathbb{R}^N$ and by Proposition 1.3.6 there exists a vector bundle E_2 such that $E_1 \oplus E_2 \cong B \times \mathbb{R}^N$ □

Another important operation on vector bundles is that of the tensor product. This is defined as follows:

Definition 1.4.14 (Tensor product of vector bundles)**.** Let $p_1 \colon E_1 \to B$ and $p_2 \colon E_2 \to B$ be two vector bundles over the same base space B. Then the *tensor product of* E_1 *and* E_2 is defined as

$$E_1 \otimes E_2 := \bigsqcup_{b \in B} p_1^{-1}(b) \otimes p_2^{-1}(b).$$

By Theorem 1.4.8 this is a vector bundle when equipped with the suitable topology.

To explicitly compute the tensor product of two vector bundles can be quite challenging, but it can be simplified if we take another point of view. To do this, we again take a step back and consider any functor $F\colon \mathfrak{V}\times\mathfrak{V}\to\mathfrak{V}$ and the corresponding vector bundle $F(E_1,E_2)$. Recall that by Proposition 1.1.10, we can view any vector bundles in terms of its gluing functions. This means the vector bundle $F(E_1,E_2)$ also admits a family of gluing functions $\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_n(\mathbb{K})\}$. These gluing functions were induced by the functions $h_\alpha\circ h_\beta^{-1}$, which in the case of $F(E_1,E_2)$ are given by

$$F(h_{1\alpha},h_{2\alpha})\circ F(h_{1\beta}^{-1},h_{2\beta}^{-1}) = F(h_{1\alpha}\circ h_{1\beta}^{-1}, h_{2\alpha}\circ h_{2\beta}^{-1}).$$

Examining the right hand side we see the functions $h_{i\alpha}\circ h_{i\beta}^{-1}\colon U_\alpha\cap U_\beta\times\mathbb{K}^n\to U_\alpha\cap U_\beta\times\mathbb{K}^n$, which in turn induce gluing functions $g_{i\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_n(\mathbb{K})$. This suggest we might be able to extend F to an operation on gluing functions, such that the gluing functions of $F(E_1,E_2)$ are given by $F(g_{1\alpha\beta},g_{2\alpha\beta})$.

Lemma 1.4.15. *Given two vector bundles E_1 and E_2 with families of gluing functions $\{g_{i\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_{n_i}(\mathbb{K})\}$, then the gluing functions for the direct sum $E_1\oplus E_2$ are given by*

$$\{g_{1\alpha\beta}\oplus g_{2\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_{n_1+n_2}(\mathbb{K})\}.$$

Proof. The gluing functions of $E_1\oplus E_2$ are induced by the functions

$$h_{1\alpha}\circ h_{1\beta}^{-1}\oplus h_{2\alpha}\circ h_{2\beta}^{-1}\colon U_\alpha\cap U_\beta\times\mathbb{K}^{n_1}\oplus\mathbb{K}^{n_2}\to U_\alpha\cap U_\beta\times\mathbb{K}^{n_1}\oplus\mathbb{K}^{n_2}.$$

Using the isomorphism $\mathbb{K}^{n_1}\oplus\mathbb{K}^{n_2}\cong\mathbb{K}^{n_1+n_2}$, we can write this in matrix form:

$$h_{1\alpha}\circ h_{1\beta}^{-1}\oplus h_{2\alpha}\circ h_{2\beta}^{-1} = \left(\mathrm{Id}, \begin{pmatrix} g_{1\alpha\beta}(b) & 0\\ 0 & g_{2\alpha\beta}(b)\end{pmatrix}\right).$$

This shows the gluing functions are indeed given by

$$\{g_{1\alpha\beta}\oplus g_{2\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_{n_1+n_2}(\mathbb{K})\}. \quad\square$$

The same discussion goes for the tensor product, and we obtain

Lemma 1.4.16. *Given two vector bundles E_1 and E_2 with families of gluing functions $\{g_{i\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_{n_i}(\mathbb{K})\}$, the gluing functions for the tensor product $E_1\otimes E_2$ are given by*

$$\{g_{1\alpha\beta}\otimes g_{2\alpha\beta}\colon U_\alpha\cap U_\beta\to GL_{n_1n_2}(\mathbb{K})\}.$$

Proof. The proof is completely analogous to the one above, but instead of taking direct sums one takes tensor products. □

This also shows the direct sum and tensor product are different operations, since given an n-dimensional vector bundle E_1 and an m-dimensional vector bundle E_2, the bundle $E_1\oplus E_2$ will be $n+m$-dimensional, whereas $E_1\otimes E_2$ will be nm-dimensional.

We know the direct sum and tensor product of vector spaces are both commutative, associative, and the tensor product is distributive with respect to the direct sum B. Viewing vector bundles in terms of its gluing functions makes it easy to verify this is also the case for the direct sum and tensor product of vector bundles.

Corollary 1.4.17. *The direct sum and tensor product of vector bundles are both, up to isomorphism, commutative, associative, and the tensor product is distributive with respect to the direct sum.*

Proof. The direct sum and tensor product of functions are both, up to isomorphism, commutative, associative, and the tensor product is distributive with respect to the direct sum, and hence by Lemma 1.4.15 and Lemma 1.4.16 this is also the case for vector bundles. □

Having an easier way to compute the tensor product of two vector bundles, we can take a look at an example.

Example 1.4.18. The set of real line bundles up to isomorphism over a given base space B, $\mathrm{Vect}^1_{\mathbb{R}}(B)$, form an abelian group using the tensor product as group operation. Given two line bundles $E_1, E_2 \in \mathrm{Vect}^1_{\mathbb{R}}(B)$, then their tensor product $E_1 \otimes E_2$ is again a line bundle since the tensor product of two one-dimensional vector bundles is again one-dimensional. The inverse of a line bundle $E_1 \in \mathrm{Vect}^1_{\mathbb{R}}(B)$ with a family of gluing functions $\{g_{1\alpha\beta} \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})\}$ functions is given by the vector bundle E_2 constructed with the inverse gluing functions $\{g_{2\alpha\beta} := g^{-1}_{1\alpha\beta} \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R})\}$. This family still satisfies the conditions stated in Lemma 1.1.9 since 1×1 matrices commute. The tensor product $E_1 \otimes E_2$ then has as gluing functions

$$g_{1\alpha\beta} \otimes g^{-1}_{1\alpha\beta} = 1 \colon U_\alpha \cap U_\beta \to GL_n(\mathbb{R}).$$

As a matter of fact, if B is compact we can equip E_1 with an inner product, and by Proposition 1.3.6 all local trivializations can be taken to be isometries. Since the gluing functions are induced by a composition of isometries, they can only take values $g_{1\alpha\beta}(x) = \pm 1$, and their squares are always equal to 1. This means every line bundle is its own inverse in $\mathrm{Vect}^1_{\mathbb{R}}(B)$.

We do not have to restrict ourselves to the real case. The complex case $\mathrm{Vect}^1_{\mathbb{C}}(B)$ can also be given a group structure. In general, the group $\mathrm{Vect}^1_{\mathbb{K}}(B)$ is called the *Picard group of B*. See ([Har77], p. 143) for the definition in the context of ringed spaces.

There are many more interesting functors on vector spaces that can be used on vector bundles, such as the dual vector space and the k-th exterior power, but throughout the rest of this thesis the main focus will lie on the direct sum and the tensor product.

2 Classification of vector bundles over S^k

In this section we will develop tools to classify vector bundles over the sphere S^k. Our main focus will be results that allow us to use ideas from algebraic topology to study vector bundles. We will be following the discussion in [Hat03].

2.1 Homotopy invariance of vector bundles

To further study vector bundles we will first be developing some technical tools. These tools are in particular needed to prove Theorem 2.1.4 and Theorem 2.2.5, which are some results relating homotopies of functions to isomorphisms of vector bundles. We start with the following two lemmas:

Lemma 2.1.1. *Let X be a compact space and $p\colon E \to X \times [a,b]$ be an n-dimensional vector bundle, which restricts to trivial bundles $p\colon E|_{X\times[a,c]} \to X \times [a,c]$, $p\colon E|_{X\times[c,b]} \to X \times [c,b]$ for some $c \in\,]a,b[$, then $p\colon E \to X \times [a,b]$ is trivial as well.*

Proof. We want to find an isomorphism $h\colon E \to X \times [a,b] \times \mathbb{K}^n$, utilizing we already have trivializations

$$\begin{aligned} h_1\colon E_1 &\to X \times [a,c] \times \mathbb{K}^n \\ h_2\colon E_2 &\to X \times [c,b] \times \mathbb{K}^n. \end{aligned}$$

To be able to define an isomorphism on E using h_1 and h_2, they must agree on $p^{-1}(X \times \{c\})$. To ensure they do, we define an isomorphism $f\colon X \times [c,b] \times \mathbb{K}^n \to X \times [c,b] \times \mathbb{K}^n$, which on each slice $X \times \{x\} \times \mathbb{K}^n$ is given by $h_1 \circ h_2^{-1}\colon X \times \{c\} \times \mathbb{K}^n \to X \times \{c\} \times \mathbb{K}^n$. Now we can define the function $h\colon E \to X \times [a,b] \times \mathbb{K}^n$ by setting

$$h(b,v)\colon = \begin{cases} h_1(b,v) & \text{if } b \in X \times [a,c] \\ f \circ h_2(b,v) & \text{if } b \in X \times [c,b] \end{cases}.$$

The function h maps each fiber by linear isomorphism and is continuous since it agrees on $p^{-1}(X \times \{c\})$. By Lemma 1.2.4 h is an isomorphism. □

Lemma 2.1.2. *Let X be a compact space and $p\colon E \to X \times I$ be a vector bundle, then there exists an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X such that each restriction $p\colon p^{-1}(U_\alpha \times I) \to U_\alpha \times I$ is trivial.*

Proof. The idea is to find suitable opens $U_i\times\,]a_i,b_i[$ over which E trivializes and then use Lemma 2.1.1 repeatedly. Since E is a vector bundle it admits an open cover $\{W_\beta\}_{\beta\in\mathcal{B}}$ over which it trivializes. Fix $x_0 \in X$, then since $\{W_\beta\}_{\beta\in\mathcal{B}}$ covers $X \times I$ it also covers $\{x_0\} \times I$. By compactness of I, we can extract a finite subcover of $\{W_\beta\}_{\beta\in\mathcal{B}}$ covering $\{x_0\} \times I$, which we relabel $\{W_i\}_{i\in\mathcal{I}}$. We assume that after projecting onto I, all W_i are open intervals. If this were not the case, we cover W_i with a finite number of opens that do satisfy this property and take those to be members of the opens cover instead. By the *Tube Lemma* ([Cra13], p. 84) we can find opens $U_i \subseteq X$ and $]a_i,b_i[\, \subseteq I$ with $U_i\times\,]a_i,b_i[\, \subseteq W_i$ such that the collection $\{U_i\times\,]a_i,b_i[\,\}_{i\in\mathcal{I}}$ covers $\{x_0\} \times I$. Without relabeling we order $\{U_i\times\,]a_i,b_i[\,\}_{i\in\mathcal{I}}$ in an increasing fashion with respect to I, and find a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of I with the property $t_i \in\,]a_{i-1},b_{i-1}[\, \cap\,]a_i,b_i[$ for all $1 \le i \le k$. By construction E is trivial over $U_i \times [t_{i-1},t_i]$ for all $i \in \mathcal{I}$. Setting $U := \bigcap_{i\in\mathcal{I}} U_i$, we see E is trivial over $U \times [t_{i-1},t_i]$ for all $i \in \mathcal{I}$ as well. Now we use Lemma 2.1.1 repeatedly on $\{U \times [t_{i-1},t_i]\}_{i\in\mathcal{I}}$ to conclude E is trivial over $U \times I$. Repeating this construction for all $x \in X$ we obtain the desired open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$. □

Having these two technical lemmas we can go on and prove the following result:

Proposition 2.1.3. *The restrictions of a vector bundle $p\colon E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is compact.*

Proof. The idea is "push along" the restriction bundle over $X \times \{1\}$ to the restriction bundle over $X \times \{0\}$. First, by Lemma 2.1.2 we can choose an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X such that E is trivial over $\{U_\alpha \times I\}_{\alpha\in\mathcal{A}}$. By compactness of X, we can find a finite subcover and we relabel this cover as $\{U_i\}_{i\in\mathcal{I}}$ and using *Urysohn's Lemma* A we find a partition of unity $\{\eta_i\}_{i\in\mathcal{I}}$ subordinate to $\{U_i\}_{i\in\mathcal{I}}$.

For $i \in \{0\} \cup \mathcal{I}$, we define functions $\varphi_i \colon X \to \mathbb{R}$ by $\varphi_i := \eta_1 + \cdots + \eta_i$. In particular, $\varphi_0 = 0$ and $\varphi_m = 1$, where $m = \max_{i \in \mathcal{I}}\{i\}$. To be able to "push along" the restriction bundle, we define

$$X_i = \{(x, \varphi_i(x)) \mid x \in X\} \subseteq X \times I.$$

Notice $X_0 = X \times \{0\}$ and $X_m = X \times \{1\}$. Now we define isomorphisms $f_i \colon E|_{X_i} \to E|_{X_{i-1}}$ for $1 \leq i \leq m$ between the restriction bundles. The isomorphisms f_i are given by

$$f_i(x, \varphi_i(x), v) = (x, \varphi_{i-1}(x), v)$$

Essentially, f_i is the identity outside $p^{-1}(U_i \times I) \cap E|_{X_i}$ and on $p^{-1}(U_i \times I) \cap E|_{X_i}$ it projects each fiber $p^{-1}(x, \varphi_i(x))$ to the fiber $p^{-1}(x, \varphi_{i-1}(x))$. This can be seen by considering a point outside U_i and computing

$$\varphi_i(x) = \varphi_{i-1}(x) + \eta_i(x) = \varphi_{i-1}(x)$$

which holds since $\operatorname{supp}(\eta_i) \subseteq U_i$. For f_i to be an isomorphism of vector bundles, we need to check it is homeomorphism and a linear isomorphism on each fiber. For continuity, we remark f_i is a composition of continuous functions. The inverse of f_i is given

$$f_i^{-1}(x, \varphi_{i-1}(x), v) = (x, \varphi_i(x), v)$$

which is continuous by the same reasoning. Outside $p^{-1}(U_i \times I) \cap E|_{X_i}$ the function f_i is the identity and thus maps fibers isomorphically to each other. On $p^{-1}(U_i \times I) \cap E|_{X_i} = p^{-1}((U_i \times I) \cap X_{i-1})$ we can use the fact that E trivializes over $U_i \times I$ which yields the trivialization $h_i \colon p^{-1}(U_i \times I) \to U_i \times I \times \mathbb{K}^n$. The composition

$$\begin{aligned} h_i \circ f_i \circ h_i^{-1} \colon \big((U_i \times I) \cap X_i\big) \times \mathbb{K}^n &\to \big((U_i \times I) \cap X_i\big) \times \mathbb{K}^n \\ (x, \varphi_i(x), v) &\mapsto (x, \varphi_{i-1}(x), v) \end{aligned}$$

is a linear isomorphism on each fiber and thus f_i must be as well. Since f_i is a homeomorphism and a linear isomorphism on each fiber, by Lemma 1.2.4 f_i is an isomorphism of vector bundles. The composition $f := f_1 \circ \cdots \circ f_m$ is again an isomorphism of vector bundles. In particular it is an isomorphism between the restrictions of E over $X_m = X \times \{1\}$ and $X_0 = X \times \{0\}$ and our result is proven. □

The above result turns out te be a very powerful tool when studying isomorphism classes of vector bundles as we will see in Theorem 2.2.5. It also implies the following result:

Theorem 2.1.4. *Given a vector bundle $p \colon E \to B$ and homotopic maps $g_0, g_1 \colon A \to B$ where A is compact, then the pullback bundles $g_0^*(E)$ and $g_1^*(E)$ are isomorphic.*

Proof. Let $G \colon A \times I \to B$ be the homotopy from g_0 to g_1. If we consider the pullback bundle $G^*(E)$, then the bundles $g_0^*(E)$ and $g_1^*(E)$ are isomorphic to the restrictions of $G^*(E)$ over $A \times \{0\}$ and $A \times \{1\}$. By Proposition 2.1.3 these bundles are isomorphic. □

Theorem 2.1.4 seems to imply there is a relation between homotopies and isomorphisms of vector bundles. We will explore this further, starting with the following corollary:

Corollary 2.1.5. *Let $g_1 \colon A \to B$ be a map of compact spaces which is a homotopy equivalence. Then the map $g_1^* \colon Vect(B) \to Vect(A)$ is a bijection of sets.*

Proof. If A and B are homotopy equivalent by g_1, then by definition there exists a function $g_2 \colon B \to A$ such that $g_2 \circ g_1 \simeq \mathrm{Id}_A$ and $g_1 \circ g_2 \simeq \mathrm{Id}_B$. Considering the functions

$$\begin{aligned} g_1^* &\colon \mathrm{Vect}^n_{\mathbb{K}}(B) \to \mathrm{Vect}^n_{\mathbb{K}}(A) \\ g_2^* &\colon \mathrm{Vect}^n_{\mathbb{K}}(A) \to \mathrm{Vect}^n_{\mathbb{K}}(B), \end{aligned}$$

we see that by the property of pullbacks and Theorem 2.1.4 $g_1^* \circ g_2^* = (g_2 \circ g_1)^* = \mathrm{Id}_A^* = \mathrm{Id}_{\mathrm{Vect}^n_{\mathbb{K}}(A)}$ and likewise that $g_2^* \circ g_1^* = \mathrm{Id}_{\mathrm{Vect}^n_{\mathbb{K}}(B)}$. This shows g_1^* is a bijection between $\mathrm{Vect}^n_{\mathbb{K}}(A)$ and $\mathrm{Vect}^n_{\mathbb{K}}(B)$ with inverse g_2^*. □

A direct application is the following example:

Example 2.1.6. Every vector bundle over a compact contractible base space is trivial.

This example motivates the studying of vector bundles over non-contractible base spaces. A nice set of base spaces is the set of spheres S^k, since they are not contractible but still well-behaved in a lot of aspects.

2.2 Clutching functions

In the previous subsection we have developed some tools for general vector bundles. Now we will restrict our attention to vector bundles over S^k. There are many reasons to study vector bundles over S^k, but one of the main reasons is that it is comparatively easy to do so. This is due to the fact S^k can be covered by two contractible opens, which can be seen by taking the northern and southern hemispheres D^k_+ and D^k_- and enlarging them slightly to open balls U_+ and U_-. Since $U_\pm$ is contractible, any vector bundle over $U_\pm$ is trivial by Example 2.1.6, which means any vector bundle over S^k can be identified with a single gluing function $g\colon U_+ \cap U_- \to GL_n(\mathbb{K})$ by Proposition 1.1.10. We can even restrict our gluing function g to the equator $S^{k-1} \subset S^k$ to obtain a function $f\colon S^{k-1} \to GL_n(\mathbb{K})$, which will yield an isomorphic vector bundle due to S^{k-1} being homotopy equivalent to $U_+ \cap U_-$ and Theorem 2.2.5. Conversely, starting with a map $f\colon S^{k-1} \to GL_n(\mathbb{K})$ we can construct a vector bundle by extending f to $U_+ \cap U_- \cong S^{k-1} \times]-\varepsilon, \varepsilon[$ by letting $f|_{S^{k-1}\times\{t\}} := f|_{S^{k-1}\times\{0\}}$ for all $t \in]-\varepsilon, \varepsilon[$. This gives rise to the following definition:

Definition 2.2.1 (Clutching function)**.** Given a map $f\colon S^{k-1} \to GL_n(\mathbb{K})$ we can construct an n-dimensional vector bundle $p\colon E_f \to S^k$ using f as gluing function as in Lemma 1.1.9, which is then called the *clutching function* for E_f.

Let us consider some examples of clutching functions:

Example 2.2.2. We start with the familiar tangent bundle TS^2. To recover the clutching function $f\colon S^1 \to GL_2(\mathbb{R})$ that corresponds to TS^2, our first task is to describe trivial bundles over $D^k_\pm$. The fact that we name the northern and southern hemisphere $D^k_\pm$ is suggestive notation for the fact there exists a homeomorphism $D^k_\pm \cong D^k$. If we flatten out D^k_+ to a disk, we can describe a trivialization of D^k_+ by taking a unit vector v_+ at the North Pole, and defining a section

$$\begin{aligned} s_{1+}\colon D^k_+ &\to D^k_+ \times \mathbb{R}^2 \\ x &\mapsto (x, v_+). \end{aligned}$$

Taking the vector v_+ and rotating it 90 degrees counter-clockwise, we obtain a vector w_+ and similarly we define the section

$$\begin{aligned} s_{2+}\colon D^k_+ &\to D^k_+ \times \mathbb{R}^2 \\ x &\mapsto (x, w_+). \end{aligned}$$

The results in the following image:

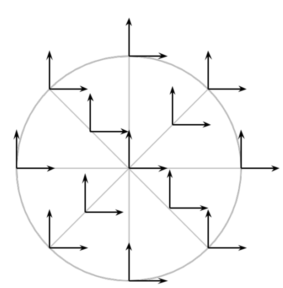

Figure 1: Flattened out D^k_+ with sections s_{1+} and s_{2+}.

If we map our flattened D^k_+ back to the sphere, these sections correspond to taking the vectors v_+ and w_+ at the North Pole and transporting them across each meridian circle, maintaining a constant angle with the meridian.

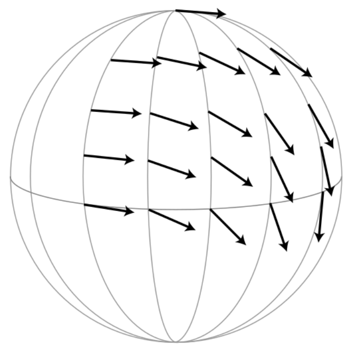

Figure 2: D^k_+ with section s_{1+} ([Hat03], p. 22).

These sections s_{1+} and s_{2+} define a trivialization of D^k_+ by demanding the map $h_+ : p^{-1}(D^2_+) \to D^2_+ \times \mathbb{R}^2$ maps the sections to the standard basis of $\mathbb{R}^2$. We can reflect the sections s_{1+} and s_{2+} across the plane containing the equator to obtain sections s_{1-}, s_{2-} and consequently a trivialization of D^k_-.

Now that we have trivializations of $D^k_\pm$, we can recover the clutching function f by reading off the coordinates of s_{1-}, s_{2-} at the equator in the coordinate system of s_{1+}, s_{2+}. Starting at a point where the two trivializations agree and going around the equator S^1 counterclockwise, f rotates s_{1-}, s_{2-} to s_{1+}, s_{2+} by an angle starting at 0 and increasing to 4π when going around the equator. If we parameterize S^1 by the angle θ from the starting point, the clutching function is then given by

$$\begin{aligned} f \colon S^1 &\to GL_2(\mathbb{R}) \\ e^{i\theta} &\mapsto \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}. \end{aligned}$$

Example 2.2.3. An example which will be of great importance in the following section is the *tautological complex line bundle*. Similar to Example 1.2.3, we can define the tautological complex line bundle H over $\mathbb{C}\mathrm{P}^1$ to be

$$p \colon H := \{(l, v) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}^2 \mid v \in l\} \to \mathbb{C}\mathrm{P}^1.$$

Notice we can view $\mathbb{C}\mathrm{P}^1$ as S^2 if we identify each equivalence class $[z_0, z_1] \in \mathbb{C}\mathrm{P}^1$ with the ratio $z := \frac{z_0}{z_1} \in \mathbb{C} \cup \{\infty\} \cong S^2$, where the identification of $\mathbb{C} \cup \{\infty\}$ with S^2 is given by the stereographic projection. If we do this, points in the disk D^2_0 inside the unit circle $S^1 \subset \mathbb{C} \cup \{\infty\} \cong \mathbb{C}\mathrm{P}^1$ can uniquely be expressed as $[\frac{z_0}{z_1}, 1] = [z, 1] \in \mathbb{C}\mathrm{P}^1$ and points in the disk D^2_∞ outside S^1 can be expressed uniquely as $[1, \frac{z_1}{z_0}] = [1, z^{-1}] \in \mathbb{C}\mathrm{P}^1$. More details about this identification will be given in Subsection 3.1.

As with the previous example, we are going to give trivializations using sections over D^2_0 and D^2_∞. We define

$$\begin{aligned} s_0 \colon D^2_0 &\to H|_{D^2_0} \\ [z, 1] &\mapsto ([z, 1], (z, 1)) \end{aligned} \qquad \begin{aligned} s_\infty \colon D^2_\infty &\to H|_{D^2_\infty} \\ [1, z^{-1}] &\mapsto ([1, z^{-1}], (1, z^{-1})). \end{aligned}$$

The sections s_0 and s_∞ define trivializations h_0 and h_∞ by requiring the second component of the sections get sent to $1 \in \mathbb{C}$. Examining their intersection S^1 we find we can pass from the trivialization over D^2_∞ to the trivialization over D^2_0 by multiplying with $z \in S^1$. This implies the clutching function for H is given by

$$\begin{aligned} f \colon S^1 &\to GL_1(\mathbb{C}) \\ z &\mapsto (z). \end{aligned}$$

Here we implicitly took D^2_∞ as D^2_+ and D^2_0 as D^2_- when identifying $\mathbb{CP}^1 \cong S^2$. If we instead chose the opposite labeling convention, where D_∞ is identified with D_- and D^2_0 is identified with D^2_+, the clutching function would have been given by $f(z) = (z^{-1})$.

What we would like to do is classify all vector bundles over S^k. Having clutching functions will make this easier, since we can identify each vector bundle with a clutching function and vice-versa. In the previous section we have seen there is a relationship between homotopic maps and isomorphic vector bundles and as it turns out this relationship also exists between homotopic clutching functions and isomorphic vector bundles over S^k, which will be made precise in Theorem 2.2.5. But first, to prove this theorem, we need the following lemma:

Lemma 2.2.4. *The space $GL_n(\mathbb{C})$ is path-connected.*

Proof. Given a matrix $A \in GL_n(\mathbb{C})$ it can be diagonalized using the elementary row operation of adding a scalar multiple of one row to another repeatedly. Adding a scalar multiple λ of row i to a different row j corresponds to multiplying A from the left with the elementary matrix

$$M = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & \lambda & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

where $m_{j,i} = \lambda$. A path from A to MA can be realized by $\gamma(t) = M_t A$ where

$$M_t = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & t\lambda & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

This path lies entirely in $GL_n(\mathbb{C})$ since $\det(M_t) = 1$ for all $t \in I$, and concatenating the paths corresponding to all elementary row operations needed to diagonalize A gives a path from A to a diagonal matrix. The set of diagonal matrices in $GL_n(\mathbb{C})$ is homeomorphic to the product

$$\underbrace{(\mathbb{C} \setminus \{0\}) \times \cdots \times (\mathbb{C} \setminus \{0\})}_{n \text{ times}}$$

which is path-connected since $\mathbb{C} \setminus \{0\}$ is path-connected and products of path-connected spaces are again path-connected. We conclude $GL_n(\mathbb{C})$ is path-connected. □

Now we are ready to state and prove the main theorem of this section:

Theorem 2.2.5. *The map $\Phi\colon [S^{k-1}, GL_n(\mathbb{C})] \to \mathrm{Vect}^n_{\mathbb{C}}(S^k)$ given by $[f] \mapsto [E_f]$ is a bijection.*

Proof. First, we must prove Φ is well-defined. Given two homotopic maps $f_0 \simeq f_1 \colon S^{k-1} \to GL_n(\mathbb{C})$, there exists a homotopy $F\colon S^{k-1} \times I \to GL_n(\mathbb{C})$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. We can use F to construct a vector bundle

$$p\colon E_F \to S^k \times I,$$

using the same reasoning as we did for the clutching functions since $D_\pm \times I$ is contractible as well. The vector bundle E_F will restrict to E_{f_0} over $S^k \times \{0\}$ and to E_{f_1} over $S^k \times \{1\}$. By Proposition 2.1.3 the bundles E_{f_0} and E_{f_1} are isomorphic since S^k is compact and we conclude Φ is well-defined.

Second, we must show Φ is a bijection and to do so we construct an inverse Ψ. Given a vector bundle $p\colon E \to S^k$, its restrictions E_+ and E_- over the upper and lower hemispheres D^k_+ and D^k_- respectively are trivial by contractibility of $D^k_\pm$ and Example 2.1.6. Choosing trivializations

$$h_\pm\colon E_\pm \to D^k_\pm \times \mathbb{C}^n$$

the composition $h_+ \circ h_-^{-1}$ induces a function

$$f\colon S^{k-1} \to GL_n(\mathbb{C})$$

as we have seen before in our discussion regarding gluing functions. We define $\Psi(E)$ to be the homotopy class of f.

We must check $\Psi(E)$ is independent of the choice of trivializations $h_\pm\colon E_\pm \to D^k_\pm \times \mathbb{C}^n$ and hence well-defined. Given different trivializations $h_{0\pm}\colon E_\pm \to D^k_\pm \times \mathbb{C}^n$ and $h_{1\pm}\colon E_\pm \to D^k_\pm \times \mathbb{C}^n$, they differ by a map $\tilde{h}_\pm\colon D^k_\pm \to GL_n(\mathbb{C})$. Since $D^k_\pm$ is contractible, $\tilde{h}_\pm$ is homotopic to a constant map

$$\begin{aligned} c_\pm\colon D^k_\pm &\to GL_n(\mathbb{C}) \\ x &\mapsto A_\pm. \end{aligned}$$

By path-connectedness of $GL_n(\mathbb{C})$, any constant map is homotopy equivalent to the map that sends everything to the identity in $GL_n(\mathbb{C})$ by composing with a path going from $A_\pm \in GL_n(\mathbb{C})$ to $\mathrm{Id} \in GL_n(\mathbb{C})$, and we obtain

$$[h_{0\pm}] = [\tilde{h}_\pm \circ h_{1\pm}] = [c_\pm \circ h_{1\pm}] = [\mathrm{Id} \circ h_{1\pm}] = [h_{1\pm}].$$

If $h_{0\pm}$ and $h_{1\pm}$ are homotopy equivalent then the compositions $h_{0+} \circ h_{0-}^{-1}$ and $h_{1+} \circ h_{1-}^{-1}$ are homotopy equivalent as well and induce homotopy equivalent clutching functions f_0 and f_1. We conclude Ψ is well-defined.

Lastly we must check Φ and Ψ are inverses of each other. This is the case since $\Psi \circ \Phi([f]) = [f]$ and $\Phi \circ \Psi([E]) = [E]$. □

We explore a direct consequence of the Theorem 2.2.5:

Corollary 2.2.6. *Every complex vector bundle over S^1 is trivial.*

Proof. The statement is equivalent to claiming $\mathrm{Vect}^n_{\mathbb{C}}(S^1)$ consists of a single element, which is the case since $[S^0, GL_n(\mathbb{C})]$ consists of a single element by path-connectedness of $GL_n(\mathbb{C})$. Theorem 2.2.5 provides a bijection between $[S^0, GL_n(\mathbb{C})]$ and $\mathrm{Vect}^n_{\mathbb{C}}(S^1)$ and we conclude $\mathrm{Vect}^n_{\mathbb{C}}(S^1)$ only consists of a single element. □

Theorem 2.2.5 made use of Lemma 2.2.4, which turned out to be of importance when finding homotopies between clutching functions. One might wonder if the same result exists for $[S^{k-1}, GL_n(\mathbb{R})]$ and $\mathrm{Vect}^n_{\mathbb{R}}(S^k)$, but this turns out not to be true since $GL_n(\mathbb{R})$ is not path-connected, which can be seen by considering the continuous surjection

$$\det(\cdot)\colon GL_n(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$$

whose image $\mathbb{R}\setminus\{0\}$ has two path components. To be more precise, $GL_n(\mathbb{R})$ has exactly two path components, namely $GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$, the matrices with positive and negative determinant respectively ([Lee00], p. 236). However, a result similar to Theorem 2.2.5 does exist for *oriented vector bundles.* To define them we must first recall the definition of an orientation of a vector space:

Definition 2.2.7 (Orientation of bases)**.** Let V be a finite-dimensional vector space and let $\{v_1, \dots v_n\}$ and $\{w_1, \dots w_n\}$ be two ordered bases, then the bases are said *to have the same orientation* if the unique linear transformation $A\colon V \to V$ mapping $v_i \mapsto w_i$ for all $1 \leq i \leq n$ has positive determinant.

Having the same orientation defines an equivalence relation on the ordered bases of a vector space V. We can use this equivalence relation to define an orientation on V:

Definition 2.2.8 (Orientation of a vector space)**.** Let V be a finite-dimensional vector space, then *an orientation of* V is an assignment of $+1$ and -1 to the equivalence classes of ordered bases of V under the equivalence relation of having the same orientation.

When considering the vector space $\mathbb{R}^n$, the standard basis is usually given positive orientation. Having the definition of an oriented vector space, we can define oriented vector bundles as:

Definition 2.2.9 (Oriented vector bundle)**.** A real vector bundle $p\colon E \to B$ is called *orientable* if each fiber can be given an orientation such that there exists an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of B such that the local trivializations $h_\alpha\colon p^{-1}(U) \to U \times \mathbb{R}^n$ carry the orientation of the fibers $p^{-1}(b)$ to the standard orientation of $\mathbb{R}^n$, and when each fiber is given such an orientation the vector bundle is called *oriented*.

Similar to before, we define:

Definition 2.2.10. The set of isomorphism classes of oriented real n-dimensional vector bundles over a base space B is denoted $\text{Vect}^n_+(B)$.

Note the isomorphisms between elements of $\text{Vect}^n_+(B)$ are required to preserve orientations. A non-example of an orientable vector bundle is the following:

Non-example 2.2.11. The Möbius bundle is non-orientable. This can be seen be noticing any orientable one-dimensional bundle over S^1 must admit a section of unit vectors which all have the same orientation, which is not the case with the Möbius bundle by Example 1.2.2.

Since all fibers of oriented vector bundles have the same orientation, the clutching function of an oriented vector bundle can be taken to map only into $GL_n^+(\mathbb{R})$. Since $GL_n^+(\mathbb{R})$ is path-connected the following holds:

Proposition 2.2.12. *There exists a bijection between* $[S^{k-1}, GL_n^+(\mathbb{R})]$ *and* $\text{Vect}^n_+(S^k)$*.*

Proof. The proof is analogous to the proof of Theorem 2.2.5. □

The power of Theorem 2.2.5 and Proposition 2.2.12 lies in the fact that we can use tools from a different area of study, algebraic topology, to analyze vector bundles.

As an example, consider real vector bundles over S^2. Given a clutching function $f\colon S^1 \to GL_n(\mathbb{R})$ it must either map entirely into $GL_n^+(\mathbb{R})$ or into $GL_n^-(\mathbb{R})$ since S^1 is path-connected. This implies every vector bundle over S^2 is orientable with two possible orientations, determined by the orientation in a single fiber. Note the same argument holds for vector bundles over S^k with $k \geq 2$, since S^{k-1} is path-connected for all $k \geq 2$. If we restrict our attention to 2-dimensional vector bundles over S^2, we can use the fact that $GL_2^+(\mathbb{R})$ deformation retracts onto $SO(2)$ by the Gram–Schmidt process D. This implies there exists a bijection

$$[S^1, GL_2^+(\mathbb{R})] \cong [S^1, SO(2)].$$

The advantage of $SO(2)$ over $GL_2^+(\mathbb{R})$ is that it is homeomorphic to S^1, since any orientation preserving isometry of $\mathbb{R}^2$ corresponds to a rotation by an angle θ. This implies, together with Proposition 2.2.12 that

$$[S^1, S^1] \cong \text{Vect}^2_+(S^2).$$

We also know any class $[f] \in [S^1, S^1]$ can be represented by the function

$$\begin{aligned} f_m\colon S^1 &\to S^1 \\ z &\mapsto z^m, \end{aligned}$$

for $m \in \mathbb{Z}$ ([Hat05], p. 29), which gives a bijection between $[S^1, S^1]$ and $\mathbb{Z}$, which in turn gives the bijection

$$\text{Vect}^2_+(S^2) \cong \mathbb{Z}.$$

Furthermore, for any vector bundle $E_{f_m} := E_m \in \mathrm{Vect}^2_+(S^2)$ there exists an isomorphism $E_m \cong E_{-m}$. This can be seen by considering the map $f_m : S^1 \to SO(2)$ under the identification $S^1 \cong SO(2)$. We then obtain

$$\begin{aligned} f_m \colon S^1 &\to SO(2) \\ e^{\theta i} &\mapsto R_{m\theta} := \begin{pmatrix} \cos(m\theta) & -\sin(m\theta) \\ \sin(m\theta) & \cos(m\theta) \end{pmatrix}. \end{aligned}$$

Recall that by Lemma 1.1.9, the vector bundles E_m and E_{-m} are given by

$$E_m = D^2_+ \times \mathbb{R}^2 \sqcup D^2_- \times \mathbb{R}^2 / (e^{i\theta}, v) \sim (e^{i\theta}, R_{m\theta} v)$$

and

$$E_{-m} = D^2_+ \times \mathbb{R}^2 \sqcup D^2_- \times \mathbb{R}^2 / (e^{i\theta}, v) \sim (e^{i\theta}, R_{-m\theta} v).$$

We claim an isomorphism between E_m and E_{-m} is given by

$$\begin{aligned} f \colon E_m &\to E_{-m} \\ (b, v) &\mapsto (b, v)\,. \end{aligned}$$

The map f is well defined on $D^2_\pm \times \mathbb{R}^2$, so we only need to check it is also well defined on the intersection $p^{-1}(S^1)$. We compute

$$\begin{aligned} f(e^{i\theta}, v) &= \left(e^{i\theta}, v\right) \\ f(e^{i\theta}, R_{m\theta} v) &= \left(e^{i\theta}, R_{m\theta} v\right). \end{aligned}$$

For f to be well defined we must have

$$E_{-m} \ni \left(e^{i\theta}, v\right) \sim \left(e^{i\theta}, R_{m\theta} v\right) \in E_{-m},$$

and this is the case since

$$\left(e^{i\theta}, R_{m\theta} v\right) \sim \left(e^{i\theta}, R_{-m\theta} R_{m\theta} v\right) = \left(e^{i\theta}, v\right).$$

Now that we know f is well defined, we remark f is also continuous and a linear isomorphism on each fiber, and hence by Proposition 1.2.4 f is an isomorphism. We conclude $E_m \cong E_{-m}$.

Since every 2-dimensional vector bundle over S^2 is orientable with at most two orientations, every element of $\mathrm{Vect}^2_{\mathbb{R}}(S^2)$ can be represented by E_m for a certain $m \geq 0$, and hence there exists a bijection

$$\mathrm{Vect}^2_{\mathbb{R}}(S^2) \cong \mathbb{N}.$$

We have seen an example of the above, namely the tangent bundle $E_2 \cong TS^2$.

We conclude this section with two lemmas about clutching functions and operations on vector bundles and an example:

Lemma 2.2.13. *Given two vector bundles $p_1 \colon E_1 \to S^k$ and $p_2 \colon E_2 \to S^k$ with clutching functions $f_1, f_2 \colon S^{k-1} \to GL_n(\mathbb{K})$ then the clutching function of $E_1 \oplus E_2$ is given by $f_1 \oplus f_2$.*

Proof. This is a direct consequence of Lemma 1.4.15, since clutching functions are a particular case of gluing functions. □

Lemma 2.2.14. *Given two vector bundles $p_1 \colon E_1 \to S^k$ and $p_2 \colon E_2 \to S^k$ with clutching functions $f_1, f_2 \colon S^{k-1} \to GL_n(\mathbb{K})$ then the clutching function of $E_1 \otimes E_2$ is given by $f_1 \otimes f_2$.*

Proof. This is a direct consequence of Lemma 1.4.16, since clutching functions are a particular case of gluing functions. □

These two lemmas allow us to simplify the analysis of vector bundles greatly, since they turn our global topological problem into a local problem about linear algebra. As an example, consider the following:

Example 2.2.15. The tautological line bundle $p\colon H \to \mathbb{C}\mathrm{P}^1$ has clutching function $f\colon S^1 \to GL_1(\mathbb{C}), z \mapsto (z)$ by Example 2.2.3. We can show there exists an isomorphism $(H \otimes H) \oplus E_0 \cong H \oplus H$ by considering their clutching functions. The clutching function for $(H \otimes H) \oplus E_0$ is given by

$$z \mapsto (f \otimes f) \oplus \mathrm{Id} = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix},$$

and the clutching function for $H \oplus H$ is given by

$$z \mapsto f \oplus f = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}.$$

We wish to construct a homotopy from $f \oplus f$ to $(f \otimes f) \oplus \mathrm{Id}$ and use Theorem 2.2.5. To do so, first notice by Lemma 2.2.4 there exists a path $\gamma\colon I \to GL_2(\mathbb{C})$ with

$$\gamma(0) = \mathrm{Id}, \quad \gamma(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Second, notice we can write

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$

A homotopy from $f \oplus f$ to $(f \otimes f) \oplus \mathrm{Id}$ is then given by

$$t \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \gamma(t) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \gamma(t).$$

We verify this is indeed a homotopy from $f \oplus f$ to $(f \otimes f) \oplus \mathrm{Id}$ by checking

$$0 \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \gamma(0) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \gamma(0) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix},$$

and

$$1 \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \gamma(1) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \gamma(1) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $f \oplus f \simeq (f \otimes f) \oplus \mathrm{Id}$, we conclude by Theorem 2.2.5 that $(H \otimes H) \oplus E_0 \cong H \oplus H$.

3 Classification of vector bundles over $\mathbb{C}\mathrm{P}^1$

In the following section we are going to restrict our attention to complex vector bundles over a single base space, $\mathbb{C}\mathrm{P}^1$. In 1957 Grothendieck showed every holomorphic vector bundle E over $\mathbb{C}\mathrm{P}^1$ splits into a sum of line bundles $E \cong \bigoplus_{i\in\mathcal{I}} E_i$ where E_i has clutching function $f_i(z) = z^{k_i}$ for some $k_i \in \mathbb{Z}$ [Gro57]. The argument involved used some big machinery from algebraic topology and geometry. In 1982 Hazewinkel and Martin gave a proof of the same fact using considerably simpler tools, namely only linear algebra [HM82]. We are going to give a similar result to [Gro57] for topological vector bundles, namely that every n-dimensional topological vector bundle E over $\mathbb{C}\mathrm{P}^1$ splits uniquely as $E \cong E_k \oplus \varepsilon^{n-1}$, where $p\colon E_k \to \mathbb{C}\mathrm{P}^1$ is a bundle with clutching function $z \mapsto z^k$ and ε^{n-1} is an $n-1$-dimensional trivial bundle. We will be using a method similar to [HM82] with the help of tools found in [Hat03].

3.1 Identification of $\mathbb{C}\mathrm{P}^1$ with S^2

Our first goal is to transfer the tools we have developed in Section 2 about vector bundles over S^k to vector bundles over $\mathbb{C}\mathrm{P}^1$. By Corollary 2.1.5 it is enough to find a homotopy equivalence between S^2 and $\mathbb{C}\mathrm{P}^1$, because homotopy equivalent base spaces give a bijection between isomorphism classes of vector bundles. As we will show, S^2 and $\mathbb{C}\mathrm{P}^1$ are not only homotopy equivalent, but even homeomorphic.

Lemma 3.1.1. *The spaces $\mathbb{C}\mathrm{P}^1$ and S^2 are homeomorphic.*

Proof. We wish to construct an explicit homeomorphism between these two spaces, which we will do by first giving a homeomorphism between $\mathbb{C}\mathrm{P}^1$ and the one-point compactification of $\mathbb{C}$, the space $\mathbb{C}\cup\{\infty\}$, and then doing the same for $\mathbb{C}\cup\{\infty\}$ and S^2. Given an equivalence class

$$[z_0, z_1] \in \{(z_0, z_1) \in \mathbb{C}^2\}/(z_0, z_1) \sim \lambda(z_0, z_1) = \mathbb{C}\mathrm{P}^1$$

we can define a homeomorphism between $\mathbb{C}\mathrm{P}^1$ and $\mathbb{C}\cup\{\infty\}$ by

$$\begin{aligned} g_1\colon \mathbb{C}\mathrm{P}^1 &\to \mathbb{C}\cup\{\infty\} \\ [z_0, z_1] &\mapsto \frac{z_0}{z_1}, \end{aligned}$$

where $\frac{z_0}{z_1} := \infty$ if $z_1 = 0$. The map g_1 is well defined since $g_1([z_0, z_1]) = \frac{z_0}{z_1} = \frac{\lambda z_0}{\lambda z_1} = g_1([\lambda z_0, \lambda z_1])$. The inverse of g_1 is given by

$$g_1^{-1}(z) = \begin{cases} [z, 1] & \text{if } |z| \le 1 \\ [1, z^{-1}] & \text{if } |z| \ge 1 \end{cases}$$

where by convention $\infty^{-1} := 0$. Both g_1 and its inverse are continuous, since g_1 is a continuous function on $\mathbb{C}$ and also continuous at ∞ by definition of the topology of a one-point compactification ([Cra13], p. 90). The inverse is continuous inside the unit circle and outside the unit circle, and on the intersection S^1 we have $[z, 1] = [z^{-1}z, z^{-1}1] = [1, z^{-1}]$ and hence the inverse is continuous as well. This makes g_1 a homeomorphism and we conclude $\mathbb{C}\mathrm{P}^1$ is homeomorphic to $\mathbb{C}\cup\{\infty\}$.

For the homeomorphism between $\mathbb{C}\cup\{\infty\}$ and S^2 we want to use the stereographic projection ([Lee00], p.30). Recall this is given by

$$\begin{aligned} g_2\colon S^2 &\to \mathbb{C}\cup\{\infty\} \\ (x_1, x_2, x_3) &\mapsto \frac{x_1 + ix_2}{1 - x_3}, \end{aligned}$$

where $(0, 0, 1) \mapsto \infty$. Again recall the inverse of g_2 is given by

$$\begin{aligned} g_2^{-1}\colon \mathbb{C}\mathrm{P}^1 &\to S^2 \\ z &\mapsto \left(\frac{\bar{z} + z}{z\bar{z} + 1}, \frac{z - \bar{z}}{i(z\bar{z} + 1)}, \frac{z\bar{z} - 1}{z\bar{z} + 1}\right). \end{aligned}$$

The stereographic projection is continuous with continuous inverse and we obtain a homeomorphism between S^2 and $\mathbb{C}\cup\{\infty\}$. Composing g_1 and g_2^{-1} we obtain a homeomorphism between $\mathbb{C}\mathrm{P}^1$ and S^2. □

Being homeomorphic implies being homotopy equivalent by simply taking the homeomorphism and its inverse as homotopy equivalences. By Corollary 2.1.5, there exists a bijection $\mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1) \cong \mathrm{Vect}^n_{\mathbb{C}}(S^2)$. This means we can use our tools from Section 2 to study vector bundles over $\mathbb{C}\mathrm{P}^1$. In particular, we obtain a bijection $[S^1, GL_n(\mathbb{C})] \cong \mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1)$ identifying each vector bundle over $\mathbb{C}\mathrm{P}^1$ with a clutching function.

3.2 Laurent polynomials

The aim of this subsection is to further simplify the study of $[S^1, GL_n(\mathbb{C})]$. As of now, the clutching functions involved were induced by trivializations of vector bundles. However, we have very little control over the behavior of the trivializations and by extension that of the clutching functions. The main part of this subsection will be devoted to showing every clutching function is homotopic to a *Laurent polynomial clutching function*.

Definition 3.2.1 (Laurent polynomial)**.** A *Laurent polynomial* is a function $l\colon U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$, given by

$$l(z) = \sum_{|n| \le N} a_n z^n,$$

with $a_n \in \mathbb{C}$.

With a *Laurent polynomial clutching function* $l\colon S^1 \to GL_n(\mathbb{C})$ we mean a function where if we consider $l(z)$ as a matrix in $GL_n(\mathbb{C})$, all entries $l_{i,j}(z)$ are Laurent polynomials. Such a matrix is called a *Laurent polynomial matrix*. Alternatively, one could regard a Laurent polynomial clutching function as a sum $\sum_{|n|\le N} A_n z^n$ where A_n are linear maps from $\mathbb{C}^n$ to itself. Note A_n need not be invertible, we only demand the sum $\sum_{|n|\le N} A_n z^n$ is invertible for all $z \in S^1$.

To show every clutching function is homotopic to a Laurent polynomial clutching function, we first need a technical lemma. The proof of this lemma does not contain any techniques relevant to the rest of this thesis, so one may skip ahead to Proposition 3.2.3.

Lemma 3.2.2. *Given a continuous function $f\colon S^1 \to \mathbb{C}$ and an $\varepsilon > 0$, there exists a Laurent polynomial $l\colon S^1 \to \mathbb{C}$ such that $|l(z) - f(z)| < \varepsilon$ for all $z \in S^1$.*

Proof. We wish to approximate f by a function $l(z) = \sum_{|n|\le N} a_n z^n$. Motivated by the Fourier series, we set

$$a_n := \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{it})\mathrm{e}^{-int}dt.$$

Note $(a_n)_{n\in\mathbb{Z}}$ is bounded since $|f|$ is a continuous functions on a compact domain and hence bounded by some $M > 0$ and we can compute

$$\begin{aligned}
|a_n| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{it})\mathrm{e}^{-int}dt\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \left|f(\mathrm{e}^{it})\mathrm{e}^{-int}\right| dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} |f(\mathrm{e}^{it})|\, |\mathrm{e}^{-int}| dt \\
&\le \frac{1}{2\pi} M \int_0^{2\pi} dt \\
&= M.
\end{aligned}$$

We conclude $|a_n| \le M$ for all $n \in \mathbb{Z}$. For $0 \le r \le 1$ we define

$$u(r, \theta) = \sum_{n\in\mathbb{Z}} a_n r^{|n|}\mathrm{e}^{in\theta}.$$

For fixed $r < 1$, the series $\sum_{n\in\mathbb{Z}} a_n r^{|n|} e^{in\theta}$ converges absolutely and uniformly in θ as can be seen by the computation

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}} a_n r^{|n|} e^{in\theta}\right| &\leq \sum_{n\in\mathbb{Z}} |a_n r^{|n|} e^{in\theta}| \\ &= \sum_{n\in\mathbb{Z}} |a_n|\,|r^{|n|}|\,|e^{in\theta}| \\ &\leq M \sum_{n\in\mathbb{Z}} |r^{|n|}| \\ &= M\left(\sum_{n=0}^{\infty} r^n + \sum_{n=1}^{\infty} r^n\right) \\ &= M\left(\frac{1}{1-r} + \frac{1}{1-r} - 1\right) \\ &= M\frac{1+r}{1-r}.\end{aligned}$$

The idea is to show u uniformly converges to f as r approaches 1. If this is the case we can sum finitely many terms of $u(r,\theta)$ with r sufficiently close to 1 to obtain the desired approximation of f by a Laurent polynomial.

First off, we fix $r < 1$ and notice

$$\begin{aligned}u(r,\theta) &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-i\pi t}dt\, r^{|n|}e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} r^{|n|} f(e^{it})e^{i\pi(\theta-t)}dt \\ &= \int_0^{2\pi} \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|} f(e^{it})e^{i\pi(\theta-t)}dt.\end{aligned}$$

The interchange of summation and integration is justified by the fact $\sum_{n=-\infty}^{\infty} r^{|n|} f(e^{it})e^{i\pi(\theta-t)}$ converges uniformly by a similar argument as seen when showing uniform convergence of $u(r,\theta)$. We define the *Poisson kernel* as

$$P(r,\varphi) := \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\varphi} \quad \text{for } 0 \leq r < 1 \text{ and } \varphi \in \mathbb{R},$$

and then $u(r,\theta) = \int_0^{2\pi} P(r,\theta - t)f(e^{it})dt$. The motivation behind defining the Poisson kernel is given by the fact that the Poisson satisfies the following properties:

1. As a function of φ, $P(r,\varphi)$ is even, periodic with period 2π, and monotone decreasing on $[0,\pi]$. In particular, $P(r,\varphi) \geq P(r,\pi) > 0$ for all $r > 1$ and $\varphi \in \mathbb{R}$.
2. For fixed $r < 1$, the integral $\int_0^{2\pi} P(r,\varphi)d\varphi = 1$.
3. For fixed $\varphi \in \left]0,\pi\right[$, $P(r,\varphi) \to 0$ as $r \to 1$.

To prove these three properties, we first compute

$$\begin{aligned}
P(r,\varphi) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}\mathrm{e}^{in\varphi} \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty} r^{n}\mathrm{e}^{in\varphi} + \frac{1}{2\pi}\sum_{n=1}^{\infty} r^{n}\mathrm{e}^{-in\varphi} \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty} (r\mathrm{e}^{i\varphi})^{n} + \frac{1}{2\pi}\sum_{n=1}^{\infty} (r\mathrm{e}^{-i\varphi})^{n} \\
&= \frac{1}{2\pi}\left(\frac{1}{1-r\mathrm{e}^{i\varphi}} + \frac{1}{1-r\mathrm{e}^{-i\varphi}} - 1\right) \\
&= \frac{1}{2\pi}\left(\frac{1-r^2}{1-r\mathrm{e}^{i\varphi}-r\mathrm{e}^{-i\varphi}+r^2}\right) \\
&= \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\varphi)+r^2}.
\end{aligned}$$

For the first property, we can use the fact $P(r,\varphi) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\varphi)+r^2}$. We know $\cos(\varphi)$ is an even function and is periodic with period 2π, so $\varphi \mapsto P(r,\varphi)$ is as well. Furthermore, $\varphi \mapsto -2r\cos(\varphi)$ is monotone increasing on $[0,\pi]$, and hence $\varphi \mapsto P(r,\varphi)$ is monotone decreasing on $[0,\pi]$. Similarly, $\varphi \mapsto P(r,\varphi)$ is monotone increasing on $[\pi, 2\pi]$ by the same reasoning, and we conclude $P(r,\varphi) \geq P(r,\pi) > 0$ for all $r > 1$ and $\varphi \in \mathbb{R}$. For the second property, we compute

$$\begin{aligned}
\int_0^{2\pi} P(r,\varphi)d\varphi &= \int_0^{2\pi}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}\mathrm{e}^{in\varphi}d\varphi \\
&= \sum_{n=-\infty}^{\infty}\frac{1}{2\pi}r^{|n|}\int_0^{2\pi}\mathrm{e}^{in\varphi}d\varphi \\
&= \sum_{n=1}^{\infty}\frac{1}{2\pi}r^{n}\int_0^{2\pi}\mathrm{e}^{in\varphi}d\varphi + \sum_{n=1}^{\infty}\frac{1}{2\pi}r^{n}\int_0^{2\pi}\mathrm{e}^{-in\varphi}d\varphi + \frac{1}{2\pi}r^0\int_0^{2\pi}\mathrm{e}^{0}d\varphi \\
&= \sum_{n=1}^{\infty}\frac{1}{2\pi}r^{n}\left[\frac{\mathrm{e}^{in\varphi}}{in}\right]_0^{2\pi} + \sum_{n=1}^{\infty}\frac{1}{2\pi}r^{n}\left[\frac{\mathrm{e}^{in\varphi}}{in}\right]_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi}d\varphi \\
&= \sum_{n=1}^{\infty}0 + \sum_{n=1}^{\infty}0 + 1 \\
&= 1,
\end{aligned}$$

and we conclude $\int_0^{2\pi} P(r,\varphi)d\varphi = 1$. For the third property, we again use $P(r,\varphi) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\varphi)+r^2}$ and take the limit for fixed $\varphi \in \,]0,\pi[$:

$$\lim_{r\to 1} P(r,\varphi) = \lim_{r\to 1}\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\varphi)+r^2} = \frac{1}{2\pi}\frac{0}{2-2r\cos(\varphi)} = 0,$$

since $2-2r\cos(\varphi) \neq 0$ for $\varphi \in \,]0,\pi[$.

We now have everything set up to show u uniformly converges to f as $r \to 1$. First, observe

$$\begin{aligned}|u(r,\theta) - f(\mathrm{e}^{i\theta})| &= \left|\int_0^{2\pi} P(r,\theta-t)f(\mathrm{e}^{it})dt - f(\mathrm{e}^{i\theta})\int_0^{2\pi} P(r,\theta-t)dt\right| \\ &= \left|\int_0^{2\pi} P(r,\theta-t)f(\mathrm{e}^{it})dt - \int_0^{2\pi} P(r,\theta-t)f(\mathrm{e}^{i\theta})dt\right| \\ &= \left|\int_0^{2\pi} P(r,\theta-t)\left(f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right)dt\right| \\ &\le \int_0^{2\pi} P(r,\theta-t)\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt,\end{aligned}$$

where we used the second and first property of the Poisson kernel in the first and last line respectively. For the final estimate, we are going to split the integral

$$I := \int_0^{2\pi} P(r,\theta-t)\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt$$

into two parts. First, note that $f\colon S^1 \to \mathbb{C}$ is a continuous function on a compact domain, and hence uniformly continuous. This entails that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right| < \varepsilon$ for $|t-\theta| < \delta$. Hence, if we only consider the integral on $[\theta-\delta, \theta+\delta]$ we can estimate

$$\begin{aligned}I_1 &:= \int_{\theta-\delta}^{\theta+\delta} P(r,\theta-t)\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt \\ &\le \int_{\theta-\delta}^{\theta+\delta} P(r,\theta-t)\varepsilon dt \\ &= \varepsilon\int_{\theta-\delta}^{\theta+\delta} P(r,\theta-t)dt \\ &\le \varepsilon\int_0^{2\pi} P(r,\theta-t)dt \\ &= \varepsilon.\end{aligned}$$

In the second-last line we used the first property of the Poisson kernel being non-negative. For the second integral, we want to integrate over $D := [0,2\pi] \setminus [\theta-\delta, \theta+\delta]$ and use the fact that $P(r,\theta-t)$ has a maximum value of $P(r,\delta)$ on $[0,2\pi] \setminus [\theta-\delta, \theta+\delta]$ due to the first property of the Poisson kernel:

$$\begin{aligned}I_2 &:= \int_D P(r,\theta-t)\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt \\ &\le \int_D P(r,\delta)\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt \\ &= P(r,\delta)\int_D \left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt \\ &= P(r,\delta)\int_0^{2\pi} \left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt.\end{aligned}$$

The integral $\int_0^{2\pi}\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt$ is uniformly bounded in θ since f is bounded. Since $\delta \in \,]0,\pi[$ we can use the third property of the Poisson kernel. This entails $P(r,\delta) \to 0$ as $r \to 1$, so by $\int_0^{2\pi}\left|f(\mathrm{e}^{it}) - f(\mathrm{e}^{i\theta})\right|dt$ being bounded we conclude $I_2 \to 0$ as $r \to 1$.

We conclude by stating

$$|u(r,\theta) - f(\mathrm{e}^{i\theta})| \le I = I_1 + I_2 \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $\varepsilon > 0$ if r is sufficiently close to 1, and hence that u converges to f uniformly in θ as r approaches 1. The function u was defined as $u(r,\theta) = \sum_{n\in\mathbb{Z}} a_n r^{|n|} \mathrm{e}^{in\theta}$, so by taking r sufficiently close to 1 and summing a sufficient finite amount of terms we obtain $|f - \sum_{|n|\leq N} a_n r^{|n|} \mathrm{e}^{in\theta}| < \varepsilon$. The Laurent polynomial that approximates f is given by $l(z) = \sum_{|n|\leq N} b_n \mathrm{e}^{in\theta}$ where $b_n := a_n r^{|n|}$. □

This technical lemma allows us to state and prove the following proposition:

Proposition 3.2.3. *Every vector bundle E_f over $\mathbb{C}\mathrm{P}^1$ with clutching function $f\colon S^1 \to GL_n(\mathbb{C})$ is isomorphic to a vector bundle E_l with Laurent polynomial clutching function $l\colon S^1 \to GL_n(\mathbb{C})$.*

Proof. The idea is to look at the space of functions $f\colon S^1 \to L_n(\mathbb{C})$, where $L_n(\mathbb{C})$ is the space of linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, and define a norm on this space. If we can show Laurent polynomials are dense in this space we can consider the subspace of functions $f\colon S^1 \to GL_n(\mathbb{C})$ and take a straight line homotopy from a function f to a Laurent polynomial l within this subspace. By Theorem 2.2.5 the vector bundles E_f and E_l will be isomorphic.

Let $C(S^1, L_n(\mathbb{C}))$ denote the vector space of continuous functions $f\colon S^1 \to L_n(\mathbb{C})$. We can endow this space with a norm by defining

$$\begin{aligned} \|\cdot\|\colon C(S^1, L_n(\mathbb{C})) &\to \mathbb{R} \\ f &\mapsto \sup_{z\in S^1} \|f(z)\|_{\sup}, \end{aligned}$$

where $\|\cdot\|_{\sup}$ is the *supremum norm* on $L_n(\mathbb{C})$, given by $A \mapsto \sup_{|v|=1} \|Av\|$. We are interested in the set of functions $C(S^1, GL_n(\mathbb{C})) \subset C(S^1, L_n(\mathbb{C}))$. The set $C(S^1, GL_n(\mathbb{C}))$ is open in $C(S^1, L_n(\mathbb{C}))$ since it is the preimage of $]0,\infty[$ under the continuous map

$$f \mapsto \inf_{z\in S^1} |\det(f(z))|.$$

Given a clutching function $f \in C(S^1, GL_n(\mathbb{C}))$, we can view $f(z)$ as a matrix with its entries being functions $f_{i,j}(z)\colon S^1 \to \mathbb{C}$. By Lemma 3.2.2, we know for every entry $f_{i,j}(z)$ and every $\varepsilon > 0$ there exists a Laurent polynomial $l_{i,j}(z)$ such that $|f_{i,j}(z) - l_{i,j}(z)| < \varepsilon$. Constructing a matrix $l(z) \in GL_n(\mathbb{C})$ with entries $l_{i,j}(z)$ uniformly approximating $f_{i,j}(z)$, we would like to show l approximates f as well. To simplify our computation, we define the *maximum norm* $\|\cdot\|_{\max} : A \mapsto \max_{i,j} |A_{i,j}|$ and remark $\|A\|_{\sup} \leq n \cdot \|A\|_{\max}$ for all $A \in L_n(\mathbb{C})$ ([HJ12], p. 365). The approximation of f by l as then shown by the following computation:

$$\begin{aligned} \|f - l\| &= \sup_{z\in S^1} \|f(z) - l(z)\|_{\sup} \\ &\leq \sup_{z\in S^1} n \cdot \|f(z) - l(z)\|_{\max} \\ &= n \left(\sup_{z\in S^1} \max_{i,j} |f_{i,j}(z) - l_{i,j}(z)| \right) \\ &= n \left(\max_{i,j} \sup_{z\in S^1} |f_{i,j}(z) - l_{i,j}(z)| \right) \\ &\leq n \left(\max_{i,j} \varepsilon \right) \\ &= n\varepsilon, \end{aligned}$$

where the interchange of supremum and maximum is justified by the maximum being a particular case of the supremum. This shows Laurent polynomial clutching functions are dense in $C(S^1, GL_n(\mathbb{C}))$.

Given a clutching function f, we can find an open ball $B(f,\varepsilon) \subseteq C(S^1, GL_n(\mathbb{C}))$ of radius ε centered at f. Because Laurent polynomial clutching functions are dense in $C(S^1, GL_n(\mathbb{C}))$, there exists a Laurent polynomial clutching function $l \in B(f,\varepsilon)$. Since the norm defined on $C(S^1, L_n(\mathbb{C}))$ satisfies the triangle inequality, we know $B(f,\varepsilon)$ is convex, thus we can take a straight line homotopy

$$(1-t)f + tl \in B(f,\varepsilon) \subseteq C(S^1, GL_n(\mathbb{C})) \text{ for all } t \in I.$$

This shows the clutching function f is homotopic to a Laurent polynomial clutching function l, and by Theorem 2.2.5 and Lemma 3.1.1 we conclude the vector bundles E_f and E_l are isomorphic. □

Proposition 3.2.3 allows us to study vector bundles over $\mathbb{C}\mathrm{P}^1$ through Laurent polynomial clutching functions, which are easier to handle than general clutching functions. We will take advantage of this in the following section.

3.3 Simplification of Laurent polynomials

Given a vector bundle $p\colon E \to \mathbb{C}\mathrm{P}^1$, we know by Proposition 3.2.3 there exists a Laurent polynomial clutching function l such that $E \cong E_l$. However, we can even further reduce the complexity of clutching functions, as will be shown in Proposition 3.3.8, which will be the main goal of this subsection. To streamline the proof of Proposition 3.3.8, we first state and prove a few lemmas.

Lemma 3.3.1. *Given a matrix A, the elementary operation of adding a scalar multiple λ of a column to another column is homotopy invariant.*

Proof. Adding a scalar multiple λ of a column to another column can be realized by post-multiplication by the matrix

$$M = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & \lambda & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where λ is in the suitable position. In the proof of Lemma 2.2.4 we have seen M is homotopic to the identity, and we conclude that for any matrix A it holds that $A \simeq AM$. □

Remark 3.3.2. Notice we did not assume anything about the field over which the matrix is given. In particular, this means this result also applies to Laurent polynomial matrices, where the scalars are given by Laurent polynomials.

Remark 3.3.3. The determinant of a matrix is left unchanged by adding a scalar multiple of a row or column to another row or column, as we have seen in the proof of Lemma 2.2.4.

Remark 3.3.4. The same proof holds for adding a scalar multiple of a row to another row, simply considering pre-multiplication by M instead of post-multiplication.

The following three results will allow us to have some control over the determinant of our Laurent polynomial clutching functions, which will be needed to prove Proposition 3.3.8.

Lemma 3.3.5. *Given a Laurent polynomial matrix $l\colon S^1 \to GL_n(\mathbb{C})$, $\det(l(z)) = \lambda z^k$ for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$.*

Proof. Consider the function $\tilde{l}$ defined as

$$\begin{aligned} \tilde{l} : \mathbb{C} \setminus \{0\} &\to \mathbb{C} \\ z &\mapsto \det\left(l\left(\frac{z}{|z|}\right)\right). \end{aligned}$$

Notice $\tilde{l}$ is a Laurent polynomial, since l is a Laurent polynomial matrix and $\det(\cdot)$ is a polynomial function. Since l maps into $GL_n(\mathbb{C})$, $\tilde{l}$ has no zeros on $\mathbb{C} \setminus \{0\}$, and hence $\tilde{l}(z) = \lambda z^k$ for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$. Since the function $z \mapsto \det(l(z))$ is the restriction of $\tilde{l}$ to S^1, we conclude $\det(l(z)) = \lambda z^k$ for some $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$. □

Corollary 3.3.6. *Given an n-dimensional vector bundle $p\colon E \to \mathbb{C}\mathrm{P}^1$ there exists a Laurent polynomial clutching function $l(z)$ with $\det(l(z)) = z^k$ for some $k \in \mathbb{Z}$ such that $E \cong E_l$.*

Proof. By the previous Lemma 3.3.5, we know there exists a Laurent polynomial clutching function $\tilde{l}(z)$ with $\det(\tilde{l}(z)) = \lambda z^k$ for some $k \in \mathbb{Z}$ such that $E \cong E_{\tilde{l}}$. The desired clutching function l is then given by

$$l(z) = \lambda^{-\frac{1}{n}} \tilde{l}(z).$$

We can verify $\det(l(z)) = \det(\lambda^{-\frac{1}{n}} \tilde{l}(z)) = \left(\lambda^{-\frac{1}{n}}\right)^n \det(\tilde{l}(z)) = \lambda^{-1}\lambda z^k = z^k$. An isomorphism between $E_{\tilde{l}}$ and E_l is given by scaling each fiber with a factor of $\lambda^{-\frac{1}{n}}$, which is continuous and linear on each fiber, hence an isomorphism by Lemma 1.2.4. The isomorphism between E and E_l is then given by composing the isomorphism $E \cong E_{\tilde{l}}$ with $E_{\tilde{l}} \cong E_l$. □

Lemma 3.3.7. *Let $A \in GL_n(\mathbb{C})$ be a matrix of the form*

$$A = \begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}$$

where $A_1 = (a_{1,1})$ is a 1×1 submatrix, A_3 is an $(n-1) \times 1$ submatrix and A_4 is an $n \times n$ square matrix, then $\det(A) = \det(A_1) \cdot \det(A_4)$.

Proof. We want to utilize *Laplace's formula* ([Lan87], p. 148), which states for any $n \times n$ square matrix B and fixed $1 \leq i \leq n$

$$\det(B) = \sum_{j=1}^{n} (-1)^{i+j} b_{i,j} M_{i,j},$$

where the *minor* $M_{i,j}$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column from B. If we fix $i = 1$ and apply this to our matrix A, we compute

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} M_{1,j} \\ &= (-1)^{1+1} a_{1,1} M_{1,1} + \sum_{j=2}^{n} (-1)^{1+j} a_{1,j} M_{1,j} \\ &= a_{1,1} \det(A_4) + \sum_{j=2}^{n} (-1)^{1+j} 0 M_{1,j} \\ &= \det(A_1) \cdot \det(A_4). \end{aligned}$$

We conclude $\det(A) = \det(A_1) \cdot \det(A_4)$. □

Having set this up, we now have the necessary tools to prove the following proposition:

Proposition 3.3.8. *Any $n \times n$ Laurent polynomial matrix $l(z)$ with $\det(l(z)) = z^k$ for some $k \in \mathbb{Z}$ is homotopic to the matrix $D_n^k\colon S^1 \to GL_n(\mathbb{C})$ given by*

$$D_n^k(z) := \begin{pmatrix} z^k & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Proof. We will give a proof by induction. For the case $n = 1$, consider a Laurent polynomial matrix $l(z) = (l_{1,1}(z))$. By assumption $\det(l(z)) = l_{1,1}(z) = z^k$, and hence $l(z) = (z^k) = D_1^k(z)$ and $\det(D_1^k(z)) = \det(l(z)) = z^k$.

Next, assume the result holds for $(n-1)\times(n-1)$ matrices. Given an $n \times n$ Laurent polynomial matrix $l(z) = (l_{i,j}(z))$ we would like to find a homotopy between l and D_n^k which preserves the determinant of l. We will do this by composing several homotopies. We start by writing $l(z) = z^m p(z)$, where $m \in \mathbb{Z}$ is taken such that $p(z)$ is a *polynomial matrix*. A polynomial matrix is defined analogously to a Laurent polynomial matrix, only with polynomial entries instead of Laurent polynomial entries.

Having a polynomial matrix $p(z)$, we can apply the Euclidean algorithm C to the first row of $p(z)$. This algorithm only adds (polynomial) scalar multiples of one entry to another, and hence the resulting matrix $\tilde{p}(z)$ is homotopic to the original polynomial matrix $p(z)$ by Lemma 3.3.1. Since adding scalar multiples of one row to another does not change the determinant, it also holds $\det(p(z)) = \det(\tilde{p}(z))$. The Euclidean algorithm ensures $\tilde{p}_{1,1}(z)$ is the greatest common divisor of the first row, and all other entries $\tilde{p}_{1,j}(z) = 0$ for $j > 1$. This gives a homotopy

$$p(z) = \begin{pmatrix} p_{1,1}(z) & p_{1,2}(z) & \dots & \dots & p_{1,n}(z) \\ p_{2,1}(z) & p_{2,2}(z) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ p_{n,1}(z) & \dots & \dots & \dots & p_{n,n}(z) \end{pmatrix} \simeq \begin{pmatrix} \tilde{p}_{1,1}(z) & 0 & \dots & \dots & 0 \\ \tilde{p}_{2,1}(z) & \tilde{p}_{2,2}(z) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \tilde{p}_{n,1}(z) & \dots & \dots & \dots & \tilde{p}_{n,n}(z) \end{pmatrix} = \tilde{p}(z)$$

By assumption we know $z^k = \det(l(z)) = \det(z^m p(z)) = z^{nm}\det(p(z))$, and hence $\det(p(z)) = z^r$ where $r = k - nm$. Since the determinant remained unchained by homotopy, we also obtain $\det(\tilde{p}(z)) = z^r$. Having applied the Euclidean algorithm, we can write

$$\tilde{p}(z) = \begin{pmatrix} \tilde{p}_1(z) & 0 \\ \tilde{p}_3(z) & \tilde{p}_4(z) \end{pmatrix},$$

where $\tilde{p}_1(z)$ is a 1×1 submatrix, $\tilde{p}_3(z)$ is a submatrix of dimension $(n-1)\times 1$ and $\tilde{p}_3(z)$ is a $(n-1)\times(n-1)$ submatrix. Utilizing Lemma 3.3.7 we obtain $z^r = \det(\tilde{p}(z)) = \det(\tilde{p}_1(z)) \cdot \det(\tilde{p}_4(z))$. Since z^r only factors into other integer powers of z and $\tilde{p}_1(z) = \tilde{p}_{1,1}(z)$, we conclude $\tilde{p}_{1,1}(z) = z^{r_1}$ for some $r_1 \in \mathbb{Z}$ and the determinant of the $(n-1)\times(n-1)$ submatrix $\tilde{p}_4(z)$ is $\det(\tilde{p}_4(z)) = z^{r_2}$, with $r_1 + r_2 = r$. Now we can apply the induction hypothesis and obtain a homotopy $\tilde{p}_4 \simeq D_{n-1}^{r_2}$. Using this homotopy, we construct the homotopy

$$\tilde{p}(z) = \begin{pmatrix} \tilde{p}_{1,1}(z) & 0 & \dots & \dots & 0 \\ \tilde{p}_{2,1}(z) & \tilde{p}_{2,2}(z) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \tilde{p}_{n,1}(z) & \dots & \dots & \dots & \tilde{p}_{n,n}(z) \end{pmatrix} \simeq \begin{pmatrix} z^{r_1} & 0 & \dots & \dots & 0 \\ \tilde{p}_{2,1}(z) & z^{r_2} & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \tilde{p}_{n,1}(z) & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Adding $\frac{\tilde{p}_{2,1}(z)}{z^{r_2}}$ times the second row and $\tilde{p}_{i,1}(z)$ times the i-th row for $3 \le i \le n$ to the first we obtain a homotopy

$$\begin{pmatrix} z^{r_1} & 0 & \dots & \dots & 0 \\ \tilde{p}_{2,1}(z) & z^{r_2} & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \tilde{p}_{n,1}(z) & 0 & \dots & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} z^{r_1} & 0 & \dots & \dots & 0 \\ 0 & z^{r_2} & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

The next homotopy will be given by one similar to the one seen in Example 2.2.15, namely a homotopy $\tilde{p}_t$

satisfying

$$\tilde{p}_0(z) := \begin{pmatrix} z^{r_1} & 0 & \dots & \dots & 0 \\ 0 & z^{r_2} & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \quad \tilde{p}_1(z) := \begin{pmatrix} z^{r_1+r_2} & 0 & \dots & \dots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

To achieve this homotopy we again use the path $\gamma\colon I \to GL_2(\mathbb{C})$ from Example 2.2.15 connecting the identity to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and analogous to Example 2.2.15 obtain a homotopy

$$\begin{pmatrix} z^{r_1} & 0 \\ 0 & z^{r_2} \end{pmatrix} \simeq \begin{pmatrix} z^{r_1+r_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Embedding this homotopy in $\tilde{p}_0$ will yield the desired homotopy $\tilde{p}_t$. Notice $\det(\tilde{p}_0(z)) = \det(\tilde{p}_1(z)) = z^{r_1+r_2} = z^r$ and $\tilde{p}_1(z) = D_n^r(z)$. Reading back the proof, we find we have a homotopy

$$l(z) = z^m p(z) \simeq z^m D_n^r(z).$$

As a sanity check, recall $r = k - nm$ and $z^k = \det(l(z)) = \det(z^m D_n^r(z)) = z^{nm} \cdot z^r = z^k$. Our final homotopy will be given by a homotopy between

$$z^m D_n^r(z) = \begin{pmatrix} z^{r+m} & 0 & \dots & \dots & 0 \\ 0 & r^m & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & r^m \end{pmatrix} \simeq D_n^k(z)$$

using a homotopy similar to $\tilde{p}_t$. This gives us the desired homotopy between l and D_n^k and we conclude $l \simeq D_n^k$ and $\det(l(z)) = \det(D_n^k(z)) = z^k$. □

Together with Corollary 3.3.6, Proposition 3.3.8 implies any n-dimensional complex vector bundle $p\colon E \to \mathbb{C}\mathrm{P}^1$ is isomorphic to a vector bundle with clutching function D_n^k, where k can be computed by considering the Laurent polynomial clutching function corresponding with E and computing its determinant. In particular, this yields the isomorphism $E \cong E_k \oplus \varepsilon^{n-1}$. This is because the clutching function of E_k is given by $z \mapsto z^k$ and the clutching function for ε^{n-1} is given by $\mathrm{Id} \in GL_{n-1}(\mathbb{C})$ and $D_n^k = (z^k) \oplus \mathrm{Id}$. From Lemma 2.2.13 and Lemma 2.2.14 we then obtain the isomorphism $E \cong E_k \oplus \varepsilon^{n-1}$.

Example 3.3.9. Revisiting Example 2.2.15 we again examine the clutching functions of $(H \otimes H) \oplus E_0$ and $H \oplus H$, given by

$$\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

respectively. Where we previously provided an explicit homotopy, we now know it suffices to compute the determinant of both clutching functions to determine of the vector bundles are isomorphic. Computing the determinants we find them to be z^2 in both cases, and we again conclude $(H \otimes H) \oplus E_0 \cong H \oplus H$.

Another example is given by

Example 3.3.10. Let E be a vector bundle over $\mathbb{C}\mathrm{P}^1$ with clutching function

$$z \mapsto \begin{pmatrix} iz & z^{-3} \\ iz^4 & z^2+1 \end{pmatrix}.$$

Classifying E would be quite difficult if we would want to find an explicit homotopy, but using Corollary 3.3.6 and Proposition 3.3.8 we know it is enough to compute the determinant:

$$\det \begin{pmatrix} iz & z^{-3} \\ iz^4 & z^2+1 \end{pmatrix} = iz(z^2+1) - iz^{-3}z^4 = iz^3 + iz - iz = iz^3.$$

By Corollary 3.3.6 we know there exists a Laurent polynomial l such that $\det(l(z)) = z^k$ for some $k \in \mathbb{Z}$ such that $E \cong E_l$, and by Proposition 3.3.8 l is homotopic to D_2^3. By Theorem 2.2.5 we conclude E is isomorphic to the vector bundle with clutching function D_2^3.

3.4 Fundamental group of $GL_n(\mathbb{C})$

So far, we have shown every n-dimensional vector bundle over $\mathbb{C}\mathrm{P}^1$ is isomorphic to a vector bundle with clutching function D_n^k, and we would like to use this to classify all vector bundles over $\mathbb{C}\mathrm{P}^1$ up to isomorphism. To do so, we first introduce an object central to algebraic topology:

Definition 3.4.1 (Fundamental group)**.** Given a topological space X and a point $x_0 \in X$, *the fundamental group of X with basepoint x_0* is the set

$$\pi_1(X, x_0) := \{\gamma \colon S^1 \to X \mid \gamma(1) = x_0\}/\simeq_{x_0},$$

where $\simeq_{x_0}$ is a basepoint preserving homotopy, with the group structure given by concatenation of loops.

In words, the fundamental group of a space X with basepoint x_0 is the group of loops in X starting and ending at x_0, up to basepoint preserving homotopy. We have already seen something similar, namely the set $[S^1, X]$ where $X = GL_n(\mathbb{K})$ in our specific case, but the difference is that we cannot endow $[S^1, X]$ with a group structure since not all loops in $[S^1, X]$ can be concatenated. However, in the particular case of $X = GL_n(\mathbb{C})$, there does exist a bijection $\varphi \colon \pi_1(GL_n(\mathbb{C}), x_0) \to [S^1, GL_n(\mathbb{C})]$, as is formalized in the following lemma:

Lemma 3.4.2. *The map $\varphi \colon \pi_1(GL_n(\mathbb{C}), x_0) \to [S^1, GL_n(\mathbb{C})]$ sending $[\gamma] \mapsto [\gamma]$ is a bijection.*

Proof. First, let us show φ is well defined. Given two representatives γ_0 and γ_1 of the same class in $\pi_1(GL_n(\mathbb{C}), x_0)$, their classes are mapped to $[\gamma_0]$ and $[\gamma_1]$ in $[S^1, GL_n(\mathbb{C})]$ respectively. The maps γ_0 and γ_1 are homotopic by a basepoint preserving homotopy, so in particular they are homotopic and hence they map to the same class in $[S^1, GL_n(\mathbb{C})]$ as well. To show injectivity, let $\varphi([\gamma_0]) = \varphi([\gamma_1]) \in [S^1, GL_n(\mathbb{C})]$. This means there exists a homotopy γ_t between γ_0 and γ_1. To ensure $[\gamma_0] = [\gamma_1] \in \pi_1(GL_n(\mathbb{C}), x_0)$, we must find a basepoint preserving homotopy. This homotopy is given by

$$(z, t) \mapsto x_0 \left(\gamma_t(1)\right)^{-1} \gamma_t(z).$$

We verify

$$(z, 0) \mapsto x_0 \left(\gamma_0(1)\right)^{-1} \gamma_0(z) = x_0 x_0^{-1} \gamma_0(z) = \gamma_0(z),$$

and similarly

$$(z, 1) \mapsto x_0 \left(\gamma_1(1)\right)^{-1} \gamma_1(z) = x_0 x_0^{-1} \gamma_1(z) = \gamma_1(z),$$

since $[\gamma_0], [\gamma_1] \in \pi_1(GL_n(\mathbb{C}), x_0)$ and hence $\gamma_0(1) = \gamma_1(1) = x_0$. Finally, we must verify the homotopy is basepoint preserving, which is the case since

$$(1, t) \mapsto x_0 \left(\gamma_t(1)\right)^{-1} \gamma_t(1) = x_0.$$

For surjectivity, let $[\gamma] \in [S^1, GL_n(\mathbb{C})]$. Since $GL_n(\mathbb{C})$ is path-connected by Lemma 2.2.4, there exists a path ν from x_0 to $\gamma(1)$. If we concatenate ν with γ and its reverse $\overline{\nu}$, we obtain a loop homotopic to γ, but based at x_0. The equivalence class of this loop is an element of $\pi_1(GL_n(\mathbb{C}), x_0)$, and gets sent to $[\gamma]$ under φ. □

Lemma 3.4.2 suggests the choice of basepoint $x_0 \in GL_n(\mathbb{C})$ does not matter. This is true more generally for path-connected spaces X since given any two basepoints $x_0, x_1 \in X$ we can find a path ν from x_0 to x_1 and pre-concatenate any loop γ with ν and then post-concatenate with its reverse $\overline{\nu}$ to move between basepoints. Due to this, if our space X is path-connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

More generally we have studied clutching functions for vector bundles over S^k, and found the bijection $[S^{k-1}, GL_n(\mathbb{K})] \cong \mathrm{Vect}^n_{\mathbb{K}}(S^k)$. Similar to $\pi_1(X)$, the homotopy groups $\pi_k(X, x_0)$ can be defined and there exist bijections $\pi_k(GL_n(\mathbb{K})) \cong [S^k, GL_n(\mathbb{K})]$. This would in turn yield bijections $\pi_k(GL_n(\mathbb{K})) \cong [S^{k-1}, GL_n(\mathbb{K})] \cong \mathrm{Vect}^n_{\mathbb{K}}(S^k)$.

Restricting our attention to $\mathbb{C}\mathrm{P}^1$ again, we would like to fully classify all vector bundles over $\mathbb{C}\mathrm{P}^1$. So far, we have shown every isomorphism class $[E] \in \mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1)$ can be identified with D^k_n, but we have yet to show D^k_n is unique. To do so, we need to show the maps $D^k_n, D^{\tilde{k}}_n \colon S^1 \to GL_n(\mathbb{C})$ are not homotopic if $k \neq \tilde{k}$. If this is the case, the representation of a vector bundle up to isomorphism by D^k_n is unique by Theorem 2.2.5.

Lemma 3.4.3. *The maps $D^k_n, D^{\tilde{k}}_n \colon S^1 \to GL_n(\mathbb{C})$ are homotopic if and only if $k = \tilde{k}$.*

Proof. If $k = \tilde{k}$, it follows by definition $D^k_n = D^{\tilde{k}}_n$ as well, and hence $D^k_n \simeq D^{\tilde{k}}_n$. For the converse, let $D^k_n \simeq D^{\tilde{k}}_n$. We wish to simplify our problem and to do we use that given two homotopic maps $f_0 \simeq f_1 \colon X \to Y$ and a third map $g \colon Y \to Z$, the compositions $g \circ f_0$ and $g \circ f_1$ our homotopic as well. If we compose D^k_n with the determinant $\det(\cdot) \colon GL_n(\mathbb{C}) \to \mathbb{C} \setminus \{0\}, A \mapsto \det(A)$, we find

$$\det(\cdot) \circ D^k_n : S^1 \to \mathbb{C} \setminus \{0\}.$$

Considering its homotopy class $[\det(D^k_n)] \in [S^1, \mathbb{C} \setminus \{0\}]$, we can use $\mathbb{C} \setminus \{0\}$ deformation retracts onto S^1, which means there exists a bijection $[S^1, \mathbb{C} \setminus \{0\}] \cong [S^1, S^1] \cong \pi_1(S^1)$. From algebraic topology we know $\pi_1(S^1) \cong \mathbb{Z}$ ([Hat05], p. 29), and under this identification $[\det(D^k_n)] = [z^k]$ will be mapped to k and $[\det(D^{\tilde{k}}_n)] = [z^{\tilde{k}}]$ will be mapped to $\tilde{k}$. Since $[D^k_n] = [D^{\tilde{k}}_n]$, it follows $[\det(D^k_n)] = [\det(D^{\tilde{k}}_n)]$ and hence $k = \tilde{k}$. □

We are now ready to state the final result:

Theorem 3.4.4. *There exist bijections $\pi_1(GL_n(\mathbb{C})) \cong \mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}$.*

Proof. Since S^2 and $\mathbb{C}\mathrm{P}^1$ are homeomorphic by Lemma 3.1.1, we obtain a bijection $\mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1) \cong \mathrm{Vect}^n_{\mathbb{C}}(S^2)$ using Corollary 2.1.5. From Theorem 2.2.5 we obtain a bijection $\mathrm{Vect}^n_{\mathbb{C}}(S^2) \cong [S^1, GL_n(\mathbb{C})]$, and $[S^1, GL_n(\mathbb{C})]$ can be identified with $\pi_1(GL_n(\mathbb{C}))$ by Corollary 3.4.2. Composing these bijections yields the bijection $\pi_1(GL_n(\mathbb{C})) \cong \mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1)$.

In Proposition 3.3.8 we have seen any isomorphism class $[E] \in \mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1)$ can be identified with a clutching function D^k_n. By Lemma 3.4.3 this identification is unique, and we can hence classify the vector bundles by the number $k \in \mathbb{Z}$, giving a bijection $\mathrm{Vect}^n_{\mathbb{C}}(\mathbb{C}\mathrm{P}^1) \cong \mathbb{Z}$. In particular, this yields the isomorphism $E \cong E_k \oplus \varepsilon^{n-1}$. □

Afterword

The final result Theorem 3.4.4 is a nice conclusion to what we have done so far, but there is much left to explore. Among other things, this thesis forms a foundation for *topological K-theory* ([Hat03], ch. 2). Given a compact pointed space X, we can define an equivalence relation on complex vector bundles over X by $E_1 \sim E_2$ if there exist trivial bundles ε^n and ε^m of dimension n and m respectively such that $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^m$. The set of equivalence classes forms an abelian group with the direct sum $\oplus$ as group operation and the trivial bundle ε^0 as identity element. This group is called the *reduced complex K-theory of X* and is denoted $\widetilde{K}(X)$. Using the tensor product $\otimes$, $\widetilde{K}(X)$ can be given a ring structure as well. We can explicitly compute the group $\widetilde{K}(X)$ for some spaces using what we have found so far, namely:

Example. For the space $S^0 \cong \{x, y\}$, $\widetilde{K}(S^0) \cong \mathbb{Z}$. The group isomorphism is given by

$$\begin{aligned} f \colon \widetilde{K}(S^0) &\to \mathbb{Z} \\ \left[\{x\} \times \mathbb{C}^k \cup \{y\} \times \mathbb{C}^{\tilde{k}}\right] &\mapsto k - \tilde{k}. \end{aligned}$$

Example. For S^1, $\widetilde{K}(S^1) \cong \{0\}$, since by Corollary 2.2.6 every complex vector bundle over S^1 is trivial.

Example. For $S^2 \cong \mathbb{C}\mathrm{P}^1$, $\widetilde{K}(S^2) \cong \mathbb{Z}$. The group isomorphism is given by

$$\begin{aligned} f \colon \widetilde{K}(S^2) &\to \mathbb{Z} \\ [D_n^k] &\mapsto k, \end{aligned}$$

which will be an isomorphism of groups by Theorem 3.4.4.

Computing the reduced complex K-theory of S^0, S^1 and S^2 we see a pattern emerging, namely $\widetilde{K}(S^n)$ is isomorphic to $\mathbb{Z}$ if n is even and to $\{0\}$ if n is odd. As it turns out, this pattern holds for all $n \in \mathbb{N}$ by the *Bott Periodicity Theorem* ([Hat03], p. 54), and a part of the proof of this powerful theorem has already been given in the proof of Proposition 3.2.3. Given a compact pointed space X, we can similarly define the *reduced real K-theory of X* by taking real vector bundles instead of complex ones, yielding the group $\widetilde{KO}(X)$. The real case of the *Bott Periodicity Theorem* states the values of $\widetilde{KO}(S^n)$ are given by

n mod 8	0	1	2	3	4	5	6	7
$\widetilde{KO}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

We can also define a similar equivalence relation, namely two vector bundles E_1 and E_2 over a compact space X being equivalent if there exists a trivial bundle ε^n such that $E_1 \oplus \varepsilon^n \cong E_2 \oplus \varepsilon^n$. This is called being *stably isomorphic* and denoted $E_1 \cong_s E_2$. The set of isomorphism classes does not form a group under the direct sum operation since the only element with an inverse is ε^0. However, this set is a monoid with respect to the direct sum $\oplus$ and can be made into a group using the *Grothendieck group construction* ([Hat03], p. 39), taking equivalence classes of formal differences of vector bundles as elements of the group. This group is denoted $KO(X)$ when considering real vector bundles and $K(X)$ when considering complex vector bundles and can also be given a ring structure using the tensor product. A nice duality between the K-theory of a space and reduced K-theory of that space is that as groups, there exist isomorphisms $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$ and $KO(X) \cong \widetilde{KO}(X) \oplus \mathbb{Z}$.

Once K-theory is set up, it becomes a powerful tool. An example of this is counting the number of linearly independent vector fields on spheres, as Adams has done in 1961 utilizing K-theory [Ada62]. Another example is proving $\mathbb{R}^n$ can only be equipped with a real division algebra structure when $n = 1, 2, 4, 8$, resulting in the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. This theorem is known as the *Hopf invariant one theorem* and was proven by Adams and Atiyah in 1964 [AA66]. The proof is comparatively easy using K-theory, and the same proof shows the only parallelizable spheres are S^1, S^3 and S^7.

A Urysohn's lemma

Partitions of unity are a tool to "go from local to global". Often times, we know a topological space has a certain property locally or we have a local construction, and we would like to extend it to the entire space using partitions of unity.

Definition A.1 (Partition of unity)**.** Given a topological space X, a partition of unity is a family of continuous functions $\{\eta_i\}_{i\in\mathcal{I}}$ where $\eta_i : X \to [0,1]$ such that for every point $x \in X$ only a finite number of η_i are non-zero and

$$\sum_{i\in\mathcal{I}} \eta_i(x) = 1.$$

If the indexing set $\mathcal{I}$ is finite, we speak of a *finite partition of unity.* If there exists an open cover $\{U_i\}_{i\in\mathcal{I}}$ of X, we might want the support of every η_i to be contained in the corresponding U_i. If this is the case, we speak of *a partition of unity subordinate to* $\{U_i\}_{i\in\mathcal{I}}$.

Given a space X and an open cover $\{U_i\}_{i\in\mathcal{I}}$, we would like to state conditions under which we are guaranteed a partition of unity subordinate to $\{U_i\}_{i\in\mathcal{I}}$. A sufficient condition turns out to be X being compact and Hausdorff, which we will show using a few definitions and results. Here, we will only be giving an overview of the results needed, sometimes providing a proof sketch. More detail and full proofs can be found in ([Cra13], ch. 5).

Definition A.2. A space X is called *normal* if it is Hausdorff and if for any two closed disjoint subsets $A, B \subset X$ there exist open sets U, V such that

$$A \subset U, B \subset V, \text{ and } U \cap V = \emptyset.$$

Being normal ensures one can separate disjoint closed subsets in X, and is an property a lot of well behaved spaces have. The following lemma will give a sufficient condition for a space X being normal:

Lemma A.3. *If a topological space X is compact and Hausdorff it is also normal.*

Proof sketch. The idea is to use the fact that any closed set inside a compact Hausdorff space is again compact. We then show that if a compact set can be separated from all points of another set, it can be separated from the set as a whole as well. Finally, we use that X is Hausdorff to separate the sets point by point and by the above obtain the desired result. □

Being normal extends from topological spaces to spaces of continuous functions, as is made precise in the following definition:

Definition A.4. Given a space X, we say the space of continuous functions from X to $\mathbb{R}$, $C(X)$ is *normal* if for an two disjoint closed subsets A, B of X there exists a function $f\colon X \to [0,1]$ such that $f|_A = 1$ and $f|_B = 0$.

These two notions of normality are related by the following result:

Lemma A.5 (Urysohn's lemma)**.** *Given a normal topological space X, the space $C(X)$ is normal as well.*

Proof sketch. The proof makes use of constructing and indexing sets using *dyadic fractions*. For a full proof, see ([Cra13], p.107). □

Now that we have the necessary background, we can go on to state results about partitions of unity:

Proposition A.6. *Let X be a topological space and let $C(X)$ be normal, then for any open cover $\{U_i\}_{i\in\mathcal{I}}$ there exists a partition of unity subordinate to $\{U_i\}_{i\in\mathcal{I}}$.*

Proof sketch. The main topological tool is a *shrinking lemma* ([Cra13], p. 103), which states for any finite open cover of X, we can find an open cover such that the closure of the opens is contained in the original open cover. If we use the shrinking lemma twice and then using the fact that $C(X)$ is normal, we can construct a partition of unity. □

Finally, we state the result most used in this thesis:

Corollary A.7. *Given a compact Hausdorff space X, for any open cover $\{U_i\}_{i\in\mathcal{I}}$ of X there exists a partition of unity subordinate to $\{U_i\}_{i\in\mathcal{I}}$.*

B The tensor product

The tensor product is an operation which, when applied to two vector spaces V and W, yields a new vector space $V \otimes W$, the *tensor product of V and W* [Sta20]. To define the tensor product we first need the following definition:

Definition B.1 (Free vector space)**.** Given a set S and a field $\mathbb{K}$, the *free vector space over S*, denoted $F(S)$, is the set of formal linear combinations

$$v = \sum_{i=1}^{n} \lambda_i s_i,$$

where $\lambda_i \in \mathbb{K}$ and $s_i \in S$, with vector addition and scalar multiplication defined as

$$\begin{aligned}(\lambda_1 s_1 + \cdots + \lambda_n s_n) + (\mu_1 s_1 + \cdots + \mu_n s_n) &= (\lambda_1 + \mu_1)s_1 + \cdots + (\lambda_n + \mu_n)s_1 \\ \mu(\lambda_1 s_1 + \cdots + \lambda_n s_n) &= \mu\lambda_1 s_1 + \cdots + \mu\lambda_n s_n.\end{aligned}$$

In summary, the free vector space over S is the vector space obtained by taking all elements of S as basis elements. Having this, we state the definition of the tensor product.

Definition B.2 (Tensor product)**.** Given vector spaces V and W over field $\mathbb{K}$, the *tensor product of V and W* is given by

$$V \otimes W := F(V \times W)/\sim,$$

where $\sim$ is the equivalence relation generated by

$$\begin{aligned}&(v,w) \sim (v,w) \\ &(v,w) \sim (v',w') \iff (v',w') \sim (v,w) \\ &(v,w) \sim (v',w'),\ (v',w') \sim (v'',w'') \implies (v,w) \sim (v',w') \\ &(v,w) + (v',w) \sim (v+v',w) \\ &\lambda(v,w) \sim (\lambda v,w),\ \lambda(v,w) \sim (v,\lambda w),\end{aligned}$$

and the equivalence class of (v,w) is denoted $v \otimes w$.

In summary, we take the free vector space $F(V \times W)$ and identify vectors with each other in an intuitive way. The first three relations are to ensure $\sim$ is indeed an equivalence relation, the last two are for arithmetic. To perform arithmetic within the tensor product choose representative elements of the equivalence class, perform the arithmetic in the usual way and then take the equivalence class of the result.

If we have bases $\{v_i\}$ of V and $\{w_j\}$ of W, a basis for $V \otimes W$ is given by $\{v_i \otimes w_j\}$. This also shows the dimension $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$, and hence ensures the tensor product is distinct from the direct sum. One of the most important properties of $V \otimes W$ is the universal property:

Lemma B.3. *Consider the map $\varphi\colon V \times W \to V \otimes W$, $(v,w) \mapsto v \otimes w$, then given a vector space X and a bilinear map $f\colon V \times W \to X$ there exists a unique bilinear map $\tilde{f}\colon V \otimes W \to X$ such that $f = \tilde{f} \circ \varphi$.*

This can be summarized in the following commutative diagram:

$$\begin{array}{ccc} V \times W & \xrightarrow{\varphi} & V \otimes W \\ & \overset{f}{\searrow} & \big\downarrow \exists! \tilde{f} \\ & & X \end{array} .$$

What this means is that for any bilinear map $f : V \times W \to X$ there is no loss of information if we consider it as a map $\tilde{f} : V \otimes W \to X$. This universal property simplifies proving properties about the tensor product, such as the tensor product being symmetric, associative, commutative and distributive with respect to the direct sum [Sta20].

C The Euclidean algorithm

The *Euclidean algorithm* is an algorithm for finding the greatest common divisor, GCD for short, of two polynomials ([Mor03], §1.2). Let $a(x)$ and $b(x)$ be two polynomials and without loss of generality assume $\deg(a(x)) \leq \deg(b(x))$. Using *polynomial long division*, we can find polynomials $q_0(x)$ and $r_0(x)$ with $\deg(r_0(x)) < \deg(b(x))$ such that

$$a(x) = q_0(x)b(x) + r_0(x).$$

Notice a polynomial $p(x)$ divides $a(x)$ and $b(x)$ if and only if it divides $b(x)$ and $r_0(x)$, and hence $\gcd(a(x), b(x)) = \gcd(b(x), r_0(x))$. We now set

$$a_1(x) = b(x), b_1(x) = r_0(x)$$

and repeat the polynomial long division to obtain new polynomials $q_1(x), r_1(x), a_2(x), b_2(x)$. Repeating this, we notice at every step

$$\deg(a_{k+1}) + \deg(b_{k+1}) < \deg(a_k) + \deg(b_k),$$

and eventually $\deg b_N(x) = 0$ for some $N \in \mathbb{N}$. This implies

$$\gcd(a, b) = \gcd(a_1, b_1) = \cdots = \gcd(a_N, 0) = a_N.$$

This algorithm can be extended to find the greatest common divisor of any finite number of polynomials by using the property that for any set of polynomials $\{p_1(x), \ldots p_n(x)\}$ it holds that

$$\gcd(p_1(x), p_2(x), \ldots, p_n(x)) = \gcd(\gcd(p_1(x), p_2(x), \ldots, p_{n-1}(x)), p_n(x)).$$

Our last remark is that $a_n(x) - q_n(x)b_n(x) = r_n(x) = b_{n+1}(x)$, or in other words that $b_{n+1}(x)$ can be computed by subtracting a polynomial multiple of $b_n(x)$ from $a_n(x)$, which is of importance when we apply the Euclidean algorithm in this thesis.

D The Gram–Schmidt process

The *Gram–Schmidt process* is way of generating a set of n orthonormal vectors from a set of n linearly independent vectors ([Lan87], p. 104). Let V be a vector space endowed with an inner product and let $\{v_1, \dots v_n\}$ be a set of n linearly independent vectors. We inductively generate a set of orthogonal vectors $\{w_1, \dots w_n\}$ by defining

$$w_1 = v_1,$$
$$w_k = v_k - \sum_{i=1}^{k-1} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i.$$

Intuitively, we are splitting the vector into all its components in terms of previously defined vectors, and removing all these components. What we are left with is a vector orthogonal to all previously defined vectors. The final step is to normalize $\{w_1, \dots w_n\}$ by defining

$$e_k = \frac{w_k}{\|w_k\|}.$$

The set $\{e_1, \dots e_n\}$ will be orthonormal, and has the property

$$< v_1, \dots v_n > = < e_1, \dots e_n > .$$

Note the entire Gram–Schmidt process is continuous, since it only consists of a composition of continuous functions.

Lastly, we would like to note the Gram–Schmidt process provides a deformation retraction of $GL_n^+(\mathbb{R})$ onto $SO(n)$. Given an invertible matrix with positive determinant $A \in GL_n^+(\mathbb{R})$, its columns span the entire space $\mathbb{R}^n$. Writing A in terms of it columns

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix},$$

we obtain a set of n independent vectors $\{v_1, \dots v_n\}$. The Gram-Schmidt process provides a way to construct an orthonormal set of vectors $\{e_1, \dots, e_n\}$ from these column vectors. Note $v_1 = \langle e_1, v_1 \rangle e_1$, $v_2 = \langle e_1, v_2 \rangle e_1 + \langle e_2, v_2 \rangle e_2$ and in general

$$v_k = \sum_{i=1}^{k} \langle e_i, v_k \rangle e_i.$$

If we let

$$Q := \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \dots & e_n \\ | & | & & | \end{pmatrix}, \quad R := \begin{pmatrix} \langle e_1, v_1 \rangle & \langle e_1, v_2 \rangle & \langle e_1, v_3 \rangle & \dots \\ 0 & \langle e_2, v_2 \rangle & \langle e_2, v_3 \rangle & \dots \\ 0 & 0 & \langle e_3, v_3 \rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then we can write $A = QR$ with $Q \in SO(n)$ being an orthogonal matrix and $R \in GL_n^+(\mathbb{R})$ being an upper triangular matrix with all entries on the diagonal being positive. This decomposition is also known as the *QR decomposition* ([Lan87], p. 284). Any matrix $A \in GL_n^+(\mathbb{R})$ can be decomposed in this matter, and a deformation retraction $F\colon GL_n^+(\mathbb{R}) \times I \to GL_n^+(\mathbb{R})$ onto $SO(n)$ is then given by

$$F(A,t) = F(QR,t) = Q \begin{pmatrix} t + (1-t)\langle e_1, v_1 \rangle & (1-t)\langle e_1, v_2 \rangle & (1-t)\langle e_1, v_3 \rangle & \dots \\ 0 & t + (1-t)\langle e_2, v_2 \rangle & (1-t)\langle e_2, v_3 \rangle & \dots \\ 0 & 0 & t + (1-t)\langle e_3, v_3 \rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We check that indeed

$$F(A,1) = F(QR,1) = Q\,\mathrm{Id} = Q \in SO(n),$$

and conclude F is a deformation retraction of $GL_n^+(\mathbb{R})$ onto $SO(n)$.

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